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Characteristic p structure of local cohomology

T E S I S

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Abstract

Given a commutative ring of characteristic $p > 0$, we can use the Frobenius morphism to study its algebraic and geometric properties. Furthermore, this map induces an action on the local cohomology modules. In this work, we study rings and modules in which this action is injective, which are called F-injective. There exists relations between F-injectivity and other F-singularities. We discuss the proof that F-injectivity is equivalent to the property that every parameter idea being Frobenius closed for Cohen-Macaulay rings. If the ring is Gorenstein and F-finite, then F-injectivity and F-purity are equivalent. We also discuss conditions in which the deformation of F-injectivity holds. In addition, we present a proof that the set of associated primes of local cohomology modules is finite for rings with F-finite representation type.

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Chapter 1

Introduction

Given a commutative ring R of characteris $p > 0$ we can study its properties via the Frobenius morphism, which sends each element to its p power. This morphism is used to detect, to classify and to measure singularities in affine varieties. Furthermore, the Frobenius map induces an action on any R-module M sending rm to $r^p m$, for every $r \in R, m \in M$. During this work we focus on the study of Frobenius properties on local cohomology groups. We say $(R, \mathfrak{m})$ is F-injective if the Frobenius action in $H^i_{\mathfrak{m}}(R)$ is injective for every i (see Section 2.2). We aim to prove relations of other F-singularities with F-injectivity. We consider reduced Noetherian rings throughout this work due to the behaviour in the singularities, for example the equivalence of F-purity and F-splitness.

Theorem A (Corollary 6.1.6, [HR76])**.** Let R be a Noetherian ring of characteristic p. Let R be F-finite. Then R is F-split if and only if R is F-pure.

In a Cohen-Macaulay ring we present a characterization of F-injectivity using Frobenius closure of ideals.

Theorem B (Corollary 5.2.3, [QS17])**.** Let $(R, \mathfrak{m}, K)$ be a local Cohen-Macaulay ring. The following are equivalent

1. Every parameter ideal is Frobenius closed,
2. There is a parameter ideal of R that is Frobenius closed.
3. R is F-injective.

In addition, when we consider a Gorenstein ring we have the equivalence between F-pure and F-injectivity.

Theorem C (Theorem 6.1.11, [Fed83])**.** Let $(R, \mathfrak{m})$ be a local reduced ring of characteristic $p > 0$ of dimension d, and F-finite. If R is F-pure, then R is F-injective. Conversely, if R is Gorenstein, F-finite and F-injective, R is F-pure.

Taking a detour from F-injectivity, we study the behaviour of the associated primes of the cohomological groups under certain conditions.

Theorem D (Theorem 4.2.3, [DQ20])**.** Let R have finite F-representation type. Then $H^t_I(R)$ has only finitely many associated primes for any ideal I and any $t \geq 0$.

Finally, deformation of F-injective ring is still an open question, however in the work of Linquan Ma and Pham Quy, we found certain conditions in which this condition holds. To study this we first focus on the deformation of F-anti-nilpotent rings and F-full rings. We have the following results regarding this topic.

Theorem E (Theorem 7.2.2, [MQ18])**.** Let $x \in R$ be a regular element. Then we have

1. If $R/(x)$ is F-anti-nilpotent, then so is R.
2. If $R/(x)$ is F-full, then so is R.

Theorem F (Corollary 7.3.9, [MQ18])**.** Let $x \in R$ be a regular element. If $R/(x)$ is strongly F-injective, then so is R.

In order to describe better these interactions we separate the content of this thesis in 6 chapters, each devoted to different aspects.

Chapter 2 is intended to be a compilation of results we use in the following chapters. We begin with several definitions of local cohomology groups. These definitions use different techniques like derived functors and Koszul cohomology. Later we discuss about local duality. We review the definition of Matlis duality, Cohen-Macaulay rings, Gorenstein rings and canonical modules. References of those results are classical notes in local cohomology and Cohen-Mcaulay rings [BH93, Hoc11, Jef18].

Our goal in Chapter 3 is talk about F-singularities mentioned before. We define the Frobenius morphism and the ring R^{1/p^e}. We enlist F-singularities we use in the next chapters and demonstrate relations among them. We devote a section to talk about F-injectivity and some results in Cohen-Macaulay rings. This chapter is based on many sources [Smi19, DQ20, QS17, MQ18, HNB17, DS16a, DSGNnB21].

In Chapter 4 we study associated primes of $H^t_I(R)$. In order to do this, we review the definition of filter regular sequences and the Nagel-Schenzel isomorphism (see [NS94]). Using the definition of finite F-representation type, we prove that the local cohomology groups have finitely many associated primes. This result is from the work of Hailong Dao and Pham Quy [DQ20] and simultaneously it was proven by Melvin Hochster and Luis Nuñéz-Betancourt [HNB17] with a different method.

Chapter 5 is devoted to prove the equivalence mentioned in Theorem B. To do this, we define Frobenius closure of ideals and we give some results regarding this F-singularity. In addition, we talk about Frobenius closure in parameter ideals which is a key to our goal. This Chapter is based on the work of Pham Quy and Kazuma Shimomoto [QS17].

Our goal in Chapter 6 is to give an example of a F-injective ring that is not F-pure (see Example 6.3.2). Whenever we have an F-finite ring, we have F-splitness is equivalent to F-purity (see Theorem A). We also prove that under certain conditions F-injective is equivalent to F-purity (see Theorem C). We prove that F-pure rings are Frobenius closed too, which proves there is no equivalence between F-purity and F-injectivity. We talk about F-anti-nilpotent rings in exact sequences. This material is from many sources [QS17, HR76, Fed83, DS16b].

In Chapter 7, we study deformation of F-injectivity. This is, if $R/(x)$ is F-injective, then R is F-injective (see Corollary 7.3.9). In order to do this, we define surjective elements which is introduced in [HMS14]. Then we prove a theorem regarding the deformation of F-full and F-anti-nilpotent rings (see Theorem E). Finally we define strongly F-injective rings and talk about its deformation (see Theorem F). This chapter is based on Linquan Ma and Pham Quy [MQ18].

Chapter 2

Background

The goal of this chapter is recalling definitions and propositions regarding local cohomology and local duality. In Section 2.1 we first define injective hulls, Matlis duality, Cohen-Macaulay modules, Gorenstein rings and canonical modules. In Section 2.2 we give four equivalent definitions of local cohomology groups along with some useful results. This chapter was based on different sources [BH93, Hoc11, Jef18]

2.1 Local Duality

The goal in this section is giving the definitions of injective resolutions, injective hull, local duality and Matlis duality. Some definitions we use are Cohen-Macaulay modules and on Gorenstein rings.

We begin with the definition of an injective module.

Definition 2.1.1. An R-module E is *injective* is for any $i : A \hookrightarrow B$ and $\varphi : A \to E$ maps of R-modules, there exists $\varphi' : B \to E$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \varphi'} \\ & & E \end{array}$$

Remark 2.1.2. The Definition 2.1.1 is equivalent to ask that any map $A \hookrightarrow B$ induces a surjection

$$\mathrm{Hom}_R(B, E) \twoheadrightarrow \mathrm{Hom}_R(A, E).$$

Hence if E is injective, the the functor $\mathrm{Hom}_R(,E)$ is exact.

Remark 2.1.3. Every R-module can be embedded in an injective module.

Due to Remark 2.1.3 we can construct a complex of injective modules.

Definition 2.1.4. A *injective resolution* of an R-module M is an exact complex of injective modules

$$E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots$$

such that $\mathrm{Ker}(E_0 \to E_1) \cong M$.

We construct the derived functor $\mathrm{Ext}^{\bullet}_{\mathrm{R}}(-,-)$ using injective resolutions

Definition 2.1.5. Let M and N be R-modules. We consider an injective resolution $E_{\bullet}$ for N. Then we define for every $i \in \mathbb{N}$

$$\mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{M},\mathrm{N}) = \frac{\mathrm{Ker}\left(\mathrm{Hom}_R\left(M,E^i\right) \to \mathrm{Hom}_R\left(M,E^{i+1}\right)\right)}{\mathrm{Im}\left(\mathrm{Hom}_R\left(M,E^{i-1}\right) \to \mathrm{Hom}_R\left(M,E^i\right)\right)}.$$

Remark 2.1.6.

1. The value of $\mathrm{Ext}^{\bullet}_{\mathrm{R}}(\cdot,\cdot)$ is independent of the resolutions chosen up to isomorphism.
2. Let $J \subseteq I \subseteq R$ be ideals. Then the map $R/J \to R/I$ induces a map in every $i \in \mathbb{N}$

$$\mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{R/I},\mathrm{M}) \to \mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{R/J},\mathrm{M}).$$

Furthermore, if

$$\cdots \subseteq I_3 \subseteq I_2 \subseteq I_1$$

is a decreasing sequence of ideals, for every $i \in \mathbb{N}$ we get a direct limit system

$$\cdots \to \mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{R/I_t},\mathrm{M}) \to \mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{R/I_{t+1}},\mathrm{M}) \to \cdots.$$

In order to define injective hulls, we need one more definition.

Definition 2.1.7. Given two R-modules $M \subseteq N$, we say that N is an *essential extension* of M is for any submodule L of N, $L \cap M \neq 0$. If $M \subseteq N$ is essential and N has no proper essential extensions, then we say that N is a *maximal essential extension.*

Now, we relate injective modules and essential extensions in the next proposition

Proposition 2.1.8. *Let $M \subseteq E$ be an R-module. Then E is injective if and only if E has no proper essential extensions. In particular, any maximal essential extension is an injective module. Furthermore, if E is injective, then all maximal essential extension of M are isomorphic to E.*

Now using Proposition 2.1.8 we define injective hulls.

Definition 2.1.9. Let M be an R-module. We say that an *injective hull* of M, denoted $\mathrm{E_R}(\mathrm{M})$ is a maximal essential extension of M.

Now we give the definition of Matlis duality.

Definition 2.1.10. Let $(R,\mathfrak{m})$ be a local ring, with $K = R/\mathfrak{m}$. The *Matlis duality functor* of R is defined as

$$(-)^{\vee} = \mathrm{Hom}_R(-,\mathrm{E_R}(\mathrm{K})).$$

Now, in order to define Cohen-Macaulay modules, we give the definition of depth of a module

Definition 2.1.11. Let $(R,\mathfrak{m})$ be a local ring and M be an R-module. The *depth* of M, denoted as $\mathrm{depth}(M)$, is the maximal length of a regular sequence on M.

We know in general $\mathrm{depth}(M) \leq \dim(M)$. The following modules are characterized by holding the equality.

Definition 2.1.12. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module. We say M is *Cohen-Macaulay* if $\operatorname{depth}(M) = \dim(M)$.

Other type of rings we are interested in are Gorenstein rings.

Definition 2.1.13. Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say R is *Gorenstein* if for every system of parameters $x_1, \ldots, x_d$ we have that

1. $x_1, \ldots, x_d$ is a regular sequence
2. $(x_1, \ldots, x_d)$ is an irreducible ideal.

Remark 2.1.14. Note that by condition 1 in the definition of Gorenstein rings, if R is Gorenstein, then R is Cohen-Macaulay.

For any module M we can get an injective resolution, therefore we have the following definition.

Definition 2.1.15. Let R be a Noetherian ring, and let M be a finitely generated R-module. The *injective dimension* of M, denoted by $\mathrm{injdim_R\,(M)}$, is defined as the length of its minimal injective resolution.

2.2 Local Cohomology

During this section R denotes a Noetherian commutative ring. We first define local cohomology groups. The first technique consists in studying derived functors. We begin with the following definition

Definition 2.2.1. Let $I \subseteq R$ an ideal and M an R-module. We define

$$\Gamma_I(M) = \{v \in M \mid I^n v = 0, \text{ for some } n \in \mathbb{N}\}$$

Proposition 2.2.2. *Let $I \subseteq R$ an ideal and M an R-module. Then Γ_I is a left exact functor in the category of R-modules.*

Proof. First we prove it is a functor. Let $\varphi : M \to N$ be a R-module homomorphism and $v \in \Gamma_I(M)$. Then there exists $n \in \mathbb{N}$ such that $I^n v = 0$, and so

$$I^n \varphi(v) = \varphi(I^n v) = 0.$$

This is, $\varphi(v) \in \Gamma_I(N)$. Consider

$$\begin{aligned} \Gamma_I(\varphi) : \Gamma_I(M) &\to \Gamma_I(N) \\ v &\mapsto \varphi(v). \end{aligned}$$

We have that $\Gamma_I(Id_M) = Id_{\Gamma_I}$. If $\varphi : M \to N$, $\phi : N \to L$ are R-module morphisms and $v \in \Gamma_I(M)$, then

$$\begin{aligned} \Gamma_I(\phi) \circ \Gamma_I(\varphi)(v) &= \Gamma_I(\phi)(\varphi(v)) \\ &= \phi \circ \varphi(v) \\ &= \Gamma_I(\phi \circ \varphi)(v). \end{aligned}$$

Now we prove that Γ_I is left exact. Let

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\phi} L \longrightarrow 0$$

be a short exact sequence. If $v \in \operatorname{Ker} \Gamma_I(\varphi)$, then $\varphi(v) = 0$, and so $v = 0$ since φ is injective. Therefore $\Gamma_I(\varphi)$ is injective. Since $\phi \circ \varphi = 0$, we have that

$$\Gamma_I(\phi) \circ \Gamma_I(\varphi) = \phi \circ \varphi = 0,$$

this is $\operatorname{Im} \Gamma_I(\varphi) \subseteq \operatorname{Ker} \Gamma_I(\phi)$. Finally, let $v \in \operatorname{Ker} \Gamma_I(\phi)$. Then $\phi(v) = 0$, and so $v \in \operatorname{Ker} \phi = \operatorname{Im} \varphi$. This is, there exists $w \in M$ such that $\varphi(w) = v$. Additionally since $v \in \Gamma_I(M)$, $I^n v = 0$ for some $n \in \mathbb{N}$. We have

$$0 = I^n v = I^n \varphi(w).$$

We conclude $\operatorname{Im} \Gamma_I(\varphi) = \operatorname{Ker} \Gamma_I(\phi)$ ■

Using this functor we give the first definition of local cohomology groups.

Definition 2.2.3. Let $I \subseteq R$ be an ideal and M an R-module. We define the i-th local cohomology group of M with respect to I by

$$H_I^i(-) = R^i \Gamma_I(-).$$

Specifically, if $M \to E^0 \to E^1 \to \dots$ is an injective resolution of M, then

$$H_I^i(M) = \frac{\operatorname{Ker}\left(\Gamma_I\left(E^i\right) \to \Gamma_I\left(E^{i+1}\right)\right)}{\operatorname{Im}\left(\Gamma_I\left(E^{i-1}\right) \to \Gamma_I\left(E^i\right)\right)}.$$

The 0-th local cohomology group of a module M is a submodule of M.

Proposition 2.2.4. *Let $I \subseteq R$ be an ideal and M an R-module. Then $H_I^0(M) \cong \Gamma_I(M)$*

Proof. Let

$$0 \to M \to E^0 \to E^1 \to \dots$$

an injective resolution for M. By Proposition 2.2.2, this induces an exact sequence

$$0 \to \Gamma_I(M) \to \Gamma_I\left(E^0\right) \to \Gamma_I\left(E^1\right) \to \dots$$

Since $E^{-1} = 0$

$$\begin{aligned} H_I^i(M) &= \frac{\operatorname{Ker}\left(\Gamma_I\left(E^i\right) \to \Gamma_I\left(E^{i+1}\right)\right)}{\operatorname{Im}\left(\Gamma_I\left(E^{i-1}\right) \to \Gamma_I\left(E^i\right)\right)} \\ &= \operatorname{Ker}\left(\Gamma_I\left(E^i\right) \to \Gamma_I\left(E^{i+1}\right)\right) \\ &= \Gamma_I(M). \end{aligned}$$

■

Another way to obtain local cohomology groups is using the two-variable functor $\operatorname{Ext}_{\mathrm{R}}^{\bullet}(\cdot,\cdot)$.

Definition 2.2.5. Let M be an R-module and let

$$\dots \subseteq I_3 \subseteq I_2 \subseteq I_1$$

be a decreasing sequence of ideals, denoted $\{I_t\}_t$. We define the *i-th local cohomology module* of M with support in I by

$$H_I^i(M) = \lim_t \mathrm{Ext}_{\mathrm{R}}^{\mathrm{i}}\left(\mathrm{R}/\mathrm{I_t}, \mathrm{M}\right).$$

Remark 2.2.6. The sequence of ideals $\{I_t\}_t$ can be replaced by either a subsequence of another decreasing sequence of ideals $\{J_t\}_t$ such that J_t and I_t are cofinal. In particular, for $I = (x_1, \ldots, x_n)$, the sequences $\{I_t\}_t$ and $\{J_t\}_t$ where $I_t = I^t$ and $J_t = (x_1^t, \ldots, x_n^t)$ are cofinal.

The following theorem gives another option in the election of replacement for the sequence of ideals.

Theorem 2.2.7. *If $I, J \subseteq R$ are ideals with the same radical, then $H_I^i(M) \cong H_J^i(M)$.*

Remark 2.2.8. Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-mod. Then $H_{\mathfrak{m}}^i(M)$ is Artinian.

Remark 2.2.9. Let $(R, \mathfrak{m})$ be a local ring and let M be an R-mod. Using

$$\operatorname{depth}_I M = \min\left\{i \mid \mathrm{Ext}_{\mathrm{R}}^{\mathrm{i}}(\mathrm{R/I}, \mathrm{M}) \neq 0\right\},$$

one can show that the first nonvanishing $H_{\mathfrak{m}}^i(M)$ occurs for $i = \operatorname{depth}_{\mathfrak{m}} M$.

We also obtain cohomology groups using the Čech complex.

Definition 2.2.10. Let $x_1, \ldots, x_d$ be a system of parameters of a ring R. We define the modified Čech complex as follows

$$\check{C}^\bullet : 0 \to \check{C}^0 \to \check{C}^1 \to \cdots \to \check{C}^d \to 0$$

where

$$\check{C}^t = \oplus_{1 \le i_1 \le \cdots \le i_t \le d} R_{x_{i_1} x_{i_2} \ldots x_{i_t}}$$

and the maps $d^t : \check{C}^t \to \check{C}^{t+1}$ are given component-wise as the localization map $R_{x_{i_1} x_{i_2} \ldots x_{i_t}} \to R_{x_{j_1} x_{j_2} \ldots x_{j_{t+1}}}$ up to a sign.

Theorem 2.2.11. *Let M be an R-mod. Then for every $i \geq 0$*

$$H_I^i(M) \cong H^i\left(M \otimes_R \check{C}^\bullet\right)$$

Remark 2.2.12. The Frobenius morphism acts on each $\check{C}^t$ from Definition 2.2.10 for every t and, since Frobenius commutes with localization, it also commutes with the differentiation maps. Therefore, F induces a morphism for every t

$$F : H_{\mathfrak{m}}^t(-) \to H_{\mathfrak{m}}^t(-).$$

Furthermore, we have

$$H_{\mathfrak{m}}^d(R) = \frac{R_{x_1 \cdots x_d}}{\sum_{i=1}^d R_{x_1 \cdots \hat{x_i} \cdots x_d}},$$

where $\hat{x_i}$ means we drop the element x_i. Thus, if $c \in H_{\mathfrak{m}}^d(R)$, then $c = \left[\frac{a}{x_1^t \cdots x_d^t}\right]$ for some $\frac{a}{x_1^t \cdots x_d^t} \in \check{C}^d$.

Now we give a third form to compute the i-th local cohomology group using the Koszul complex.

Definition 2.2.13. Let R be a ring. Given a sequence of n elements $\underline{x} = x_1, \ldots, x_n$ we define the *cohomological Koszul complex* inductively as follows

- $$K^\bullet(x_1, R): \qquad 0 \longrightarrow K_1 \xrightarrow{\cdot x_1} K_0 \longrightarrow 0$$

- $K^\bullet(\underline{x}, R) : K^\bullet(x_1, R) \otimes \cdots \otimes K^\bullet(x_n, R)$.

If M is an R-module, we define
$$K^\bullet(\underline{x}, M) = K^\bullet(\underline{x}, R) \otimes M.$$

Remark 2.2.14. From the definition of the cohomological Koszul complex, we can form the complex
$$H^\bullet(\underline{x}^\infty, M) := \lim_t K^\bullet\left(\underline{x}^t, M\right).$$

This definition is also equivalent to the previous one.

Theorem 2.2.15. *Let $I = (x_1, \ldots, x_n) \subset R$ be an ideal and M be an R-module. Then*
$$H_I^\bullet(M) \cong H^\bullet(\underline{x}^\infty, M).$$

Remark 2.2.16. For $I = (x_1, \ldots, x_n)$ and ideal in R, we have that
$$H_I^n(R) \cong \lim_t \frac{R}{(x_1^t, \ldots, x_n^t)}.$$

The following theorem states equivalent conditions for Gorenstein rings using local cohomology groups.

Theorem 2.2.17. *Let $(R, \mathfrak{m})$ be a local ring of dimension d. The following are equivalent.*

1. *R is Gorenstein.*
2. *R is Cohen-Macaulay and some system of parameters generates an irreducible ideal.*
3. $$\mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{K}, \mathrm{E}) \cong \begin{cases} 0 & i < d \\ K & i = d. \end{cases}$$
4. *R is Cohen-Macaulay and $H_{\mathfrak{m}}^d(R) \cong E_R(K)$.*
5. $\mathrm{injdim}_{\mathrm{R}}(\mathrm{R}) < \infty$.
6. $\mathrm{injdim}_{\mathrm{R}}(\mathrm{R}) = d$.

Local cohomology groups are used to define generalized Cohen-Macaulay rings.

Definition 2.2.18. Let $(R, \mathfrak{m})$ be a local ring of dimension d. We say R is *generalized Cohen-Macaulay ring* if for every $i = 0, \ldots, i-1$ the length of $H_{\mathfrak{m}}^i(R)$ is finite.

There is a module on Cohen-Macaulay rings strongly related with local cohomology.

Definition 2.2.19. A *canonical module* over a Cohen-Macaulay ring $(R, \mathfrak{m})$ is a finitely generated module denoted by ω_R such that
$$\mathrm{Hom}_R(\omega_R, \mathrm{E}_{\mathrm{R}}(\mathrm{K})) \cong H_{\mathfrak{m}}^{\dim(R)}(R).$$

Remark 2.2.20. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. By Theorem 2.2.17, R is Gorenstein if and only if
$$H_{\mathfrak{m}}^{\dim(R)}(R) \cong \mathrm{E}_{\mathrm{R}}(\mathrm{K}) \cong \mathrm{Hom}_R(R, \mathrm{E}_{\mathrm{R}}(\mathrm{K})).$$
Hence, R is a canonical module if and only if R is Gorenstein.

Finally, we have apply Matlis duality in Cohen Macaulay rings.

Theorem 2.2.21. *If $(S, \mathfrak{m}) \to (S^*, \mathfrak{n})$ is a local homomorphism of Cohen-Macaulay local rings such that S^* is module-finite over S, and $h = \dim S - \dim S^*$, then*

$$\omega_{S^*} \cong \mathrm{Ext}^{\mathrm{h}}_{\mathrm{S}}\left(\mathrm{S}^*, \mathrm{S}\right).$$

In particular, if S is Gorenstein and S^* is local, Cohen-Macaulay, and module-finite over S, then

$$\omega_{S^*} \cong \mathrm{Hom}_S\left(S^*, S\right).$$

Finally, we conclude with the local duality theorem.

Theorem 2.2.22. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d, and M be an R-module. Then*

1. *if M is finitely generated as R-module, then $H^i_{\mathfrak{m}}(M) = \mathrm{Ext}^{\mathrm{d\text{-}i}}_{\mathrm{R}}(\mathrm{M}, \mathrm{R})^{\vee}$;*
2. *if R is complete, then $H^i_{\mathfrak{m}}(M)^{\vee} = \mathrm{Ext}^{\mathrm{d\text{-}i}}_{\mathrm{R}}(\mathrm{M}, \mathrm{R})$.*

Chapter 3

Methods in prime characteristic

Through this chapter we consider R a Noetherian ring of prime characteristic $p > 0$. Our goal in Section 3.1 is to introduce the Frobenius morphism. In Section 3.2 we give the several definitions, and relations we use in following chapters. The main topic in Section 3.3 is F-injectivity which is the central topic in this thesis. This chapter is based on recent papers [DS16a], [QS17], [DQ20], [MQ18], [DSGNnB21], [HNB17].

3.1 Frobenius morphism

We begin this section with the definition of the Frobenius morphism. Later we present some properties of this map.

Definition 3.1.1. We define the *Frobenius morphism* F by

$$\begin{aligned} F : R &\to R \\ r &\mapsto r^p \end{aligned}$$

The *iterated Frobenius morphism* is the map F composed e times with itself. It is denoted by F^e.

Along with the Frobenius morphisms we obtain new ideals in R.

Definition 3.1.2. Let $I \subseteq R$ an ideal. We denote $I^{[p^e]}$ the ideal generated by the p^e-powers of the elements of I.

Being reduced is equivalent to having an injective Frobenius map. For this reason, during this manuscript we often work with reduced rings.

Proposition 3.1.3. *Let R be a ring. The Frobenius morphism is injective if and only if R is reduced.*

Proof. Let R be reduced. Then F has to be injective, because $x^p = 0$ if and only if $x = 0$.

Now, suppose that the Frobenius map is injective. We proceed by contradiction. Let $x \in R - \{0\}$ be a nilpotent element. Then there exists $\alpha \in \mathbb{N}$ such that $x^\alpha = 0$. In addition, we can find an element $e \in \mathbb{N}$ such that $\alpha < p^e$.

Since F is injective, we get that F^e is also injective. Thus, $F^e(x) = x^{p^e} = 0$, which is a contradiction. ∎

When R is reduced, we define an R-module used during this work.

Definition 3.1.4. Let R be a reduced ring. Let $P_1, \ldots, P_l$ be the minimal prime ideals of R. We define the ring R^{1/p^e} by

$$R^{1/p^e} = \left\{ x \in \bigoplus_{i=1}^{l} \overline{\mathrm{Frac}\,(R/P_i)} \mid x^{p^e} \in R \right\}$$

Remark 3.1.5. Suppose that R is reduced and M is an R-module. We have that

$$\frac{R}{I} \otimes M^{1/p} \cong \frac{M^{1/p}}{IM^{1/p}} \cong \left(\frac{M}{I^{[p]}M} \right)^{1/p}$$

Remark 3.1.6. Note that we have the following ring chain

$$R \subseteq R^{1/p} \subseteq R^{1/p^2} \subseteq R^{1/p^3} \subseteq \cdots$$

Definition 3.1.7. We define the *perfect closure* of a reduced ring R by

$$R^{\infty} = \cup_{e \in \mathbb{N}} R^{1/p^e}$$

The idea of the Frobenius map can be extended to modules.

Definition 3.1.8. Let $(R, \mathfrak{m})$ be a local ring. A *Frobenius action* on an R-module M, is an additive map $F : M \to M$ that for all $u \in M$ and $r \in R$,

$$F\,(ru) = r^p F\,(u)$$

Furthermore, the Frobenius map induces a functor in the category of R-modules.

Remark 3.1.9. There exists a equivalence of categories between R-modules and $R^{1/p}$-modules. Let G be the functor given by this equivalence going from R-modules to $R^{1/p}$-modules. Consider the the map

$$\begin{aligned} \gamma : \quad \mathrm{mod}\ (R) &\to \quad \mathrm{mod}\ (R) \\ M &\mapsto M \otimes_R R^{1/p}. \end{aligned}$$

The composition $\mathfrak{F} = G^{-1} \circ \gamma$ is called *Peskine-Szpiro's functor.*

Using the Frobenius morphism we construct another isomorphic ring

Remark 3.1.10. Consider the set

$$\mathrm{End}\,(\mathrm{R}) = \{\varphi : R \to R \mid \varphi \text{ is a ring homomorphism}\}\,.$$

Note that for any $r \in R$, the map $\cdot r$ is in $\mathrm{End}\,(\mathrm{R})$. In addition, the Frobenius map is also in $\mathrm{End}\,(\mathrm{R})$.

Definition 3.1.11. We define the ring non-commutative R-algebra generated by the map Frobenius F with relations $r^p F = Fr$, for every $r \in R$, this is

$$R\,\{F\} = \frac{R\,\langle F \rangle}{R\,(r^p F - Fr \mid r \in R)}.$$

Remark 3.1.12. Note that M is a left $R\,\{F\}$*-module* if and only if M has a Frobenius action. Hence the finite direct sum of finite generated left $R\,\{F\}$-modules is also a $R\,\{F\}$-module.

3.2 F-singularities

In this section we review some F-singularities we use in the next chapters. In addition, we give some results about their relations and properties. During this section R represents a Noetherian reduced ring of characteristic $p > 0$.

We begin with some of definitions. The first one is also called *F-compatible*.

Definition 3.2.1. We say that an R-module M is *F-stable* if $F(M) \subseteq M$

Definition 3.2.2. We say that the Frobenius action on an R-module M is *nilpotent* if $F^e(M) = 0$ for some $e > 0$.

Definition 3.2.3. Let M be an R-module with a Frobenius action F. We say the Frobenius action on M is *full* if the map

$$\mathfrak{F}^e(M) \to M$$

is surjective for some $e \geq 1$ (equivalently for every $e \geq 1$) where $\mathfrak{F}$ denote the Peskine-Szpiro's functor.

Remark 3.2.4. The Definition 3.2.3 is equivalent to

$$RF_R^e(M) = M$$

for every $e \geq 1$.

Definition 3.2.5. We say that the Frobenius action on an R-module M is *anti-nilpotent* if for any F-stable submodule $N \subseteq M$, the induced Frobenius action F on M/N is injectively.

Remark 3.2.6. If M is an anti-nilpotent R-module, then the Frobenius action is injective on M.

There is a strong relation between F-stable and anti-nilpotent modules.

Lemma 3.2.7. *Let M be an R-module. The Frobenius action on M is an anti-nilpotent if and only if every F-stable submodule $N \subseteq M$ is full. In particular, if M is anti-nilpotent, then M is full.*

Proof. First, suppose M is anti-nilpotent. Let $N \subseteq M$ be a F-stable submodule. Then $F(N) \subseteq N$. Let $N' = F(N)R$. By definition, $N' \subseteq N$ is an F-stable submodule of M.

If $N' \subsetneq N$, there exists $u \in N \setminus N'$. Then $F(u) \in N'$. Hence $F(u) = 0$ in M/N'. Since M is nilpotent, F is injective in M/N'. Therefore $u \in N'$, which is a contradiction. We conclude $N = N'$

Now, suppose every F-stable submodule of M is full, and let $N \subseteq M$ be F-stable submodule such that F is not injective on M/N. There exists $y \notin N$ such that $F(y) \in N$. Let $N'' = N + R(y)$. We have that

$$F(N'') = F(N) + F(R(y)) \subseteq N \subsetneq N''.$$

This is, $N'' \subseteq M$ is F-stable, and so full. However, $F(N'') \subseteq N \subsetneq N''$, which is a contradiction. ∎

Remark 3.2.8. Let M be an R-module endowed with a Frobenius action F. Then for every $r \in R$, the map rF denotes another Frobenius action. Moreover, is rF is full or anti-nilpotent, then so is F.

However, a full ring is not necessarily anti-nilpotent.

Example 3.2.9. Let $I = (x_1, \ldots, x_t)$ be an ideal in R. We have that

$$H_I^t(R) = \frac{R_{x_1 \cdots x_t}}{Im\left(\oplus_{i=1}^t R_{x_1 \cdots \hat{x_i} \cdots x_t} \to R_{x_1 \cdots x_t}\right)}.$$

Furthermore the Frobenius action on H_I^t is given by $\frac{1}{x_1 \cdots x_t} \mapsto \dfrac{1}{x_1^p \cdots x_t^p}$. Note that

$$R^{1/p^e} \otimes H_I^t(R) = \left[H_I^t(R)\right]^{1/p^e}.$$

Hence, for $[l] \in H_I^t(R)$ there exists $[l]^{1/p^e} \in \left[H_I^t(R)\right]^{1/p^e}$ such that $F^e\left([l]^{1/p^e}\right) = ([l])$. Therefore, the Frobenius action on $H_I^t(R)$ is full.

However, for $R = K[\![x, y]\!]$ and $I = (x)$, we have that

$$H_{(x)}^1(R) \cong K[\![y]\!]_x \oplus \cdots \oplus K[\![y]\!]_{x^n} \oplus \cdots.$$

Let N be the submodule generated by $\left\{y^2 x^{-n}\right\}_{n=1}^{\infty}$. Note that

$$F\left(y^2 x^{-n}\right) = y^{2p} x^{-np} \in N,$$

hence N is F-stable. On the other hand, $F\left(yx^{-1}\right) = y^p x^{-p} \in N$ but $yx^{-1} \notin N$ and so the Frobenius action on $H_{(x)}^1 R/N$ is not injective.

We now continue with some definitions.

Definition 3.2.10. We say that R has *finite F-representation type* by finitely generated R-modules $M_1, \ldots, M_s$ if for every $e \geq 0$, there exist $n_{e,1}, \ldots, n_{e,s} \in \mathbb{Z}$ such that

$$R^{1/p^e} \cong \oplus_{i=1}^s M_i^{n_{e,i}}.$$

Definition 3.2.11. We say that R is *F-finite* if it is a finitely generated R-module via F.

Remark 3.2.12. Finite F-representation type implies F-finite.

Definition 3.2.13. The ring R is called *F-pure* if the Frobenius endomorphism is a pure map. This is for any R-module M

$$R \otimes M \longrightarrow R \otimes M$$

is injective.

Example 3.2.14. Examples of F-pure rings are regular local rings, Stanley–Reisner rings, finitely generated normal semigroup rings and determinantal rings.

Definition 3.2.15. Let R be a domain. We say that R is *Frobenius split*, or *F-split* if there is a map

$$\varphi : R^{1/p} \to R$$

such that $\varphi \circ F = Id_R$.

Remark 3.2.16. Saying that R is F-split is equivalent to the following:

- there exists $\pi \in \operatorname{Hom}\left(R^{1/p}, R\right)$ such that $\pi\left(1^{1/p}\right) = 1$, and
- $R^{1/p} \cong R \oplus M$, with M an R-module.

We mention the equivalence of F-purity and F-splitness. However, it follows from results in Chapter 6.

Theorem 3.2.17. *If R is F-finite, then being F-pure is equivalent to being F-split.*

Definition 3.2.18. We say that a local ring $(R, \mathfrak{m})$ is F-*full* if the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is full for every $i \geq 0$.

Remark 3.2.19. Definition 3.2.18 is equivalent to say that R is F-full if the map

$$R^{1/p} \otimes H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}\left(R^{1/p}\right)$$

is surjective for every i.

Now, F-fullness localizes.

Proposition 3.2.20. *Let $(R, \mathfrak{m})$ be an F-finite and F-full local ring. Then R_P is also F-full for every $P \in \operatorname{Spec} R$.*

Proof. By a result from Gabber [[Gab04], Remark 13.6]. $R = A/I$ for some regular ring A and $I \subseteq A$ ideal. Let n be the dimension of A. Since R is F-full we have the surjective map

$$R^{1/p} \otimes H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_{\mathfrak{m}}\left(R^{1/p}\right).$$

Using the Matlis duality functor and the tensor-hom adjunction we have

$$H^i_{\mathfrak{m}}\left(R^{1/p}\right)^{\vee} \hookrightarrow \operatorname{Hom}\left(R^{1/p} \otimes H^i_{\mathfrak{m}}(R), E_R(K)\right) \quad \cong \quad \operatorname{Hom}\left(R^{1/p}, \operatorname{Hom}\left(H^i_{\mathfrak{m}}(R), E_R(K)\right)\right).$$

By Theorem 2.2.22, we have that

$$\mathrm{Ext}^{\mathrm{n\text{-}i}}_{\mathrm{R}}\left(\mathrm{A}, \mathrm{R}^{1/\mathrm{p}}\right) \hookrightarrow \operatorname{Hom}\left(R^{1/p}, \mathrm{Ext}^{\mathrm{n\text{-}i}}_{\mathrm{R}}\left(\mathrm{A}, \mathrm{R}^{1/\mathrm{p}}\right)\right).$$

Hence, for every $Q \in \operatorname{Spec} R$, we have the injective map

$$\mathrm{Ext}^{\mathrm{n\text{-}i}}_{\mathrm{R}}\left(\mathrm{A}_{\mathrm{Q}}, \mathrm{R}^{1/\mathrm{p}}_{\mathrm{Q}}\right) \hookrightarrow \operatorname{Hom}\left(R^{1/p}_Q, \mathrm{Ext}^{\mathrm{n\text{-}i}}_{\mathrm{R}}\left(\mathrm{A}_{\mathrm{Q}}, \mathrm{R}^{1/\mathrm{p}}_{\mathrm{Q}}\right)\right).$$

Using Theorem 2.2.22 and tensor-hom adjunction

$$H^i_{\mathfrak{m}}\left(R^{1/p}_Q\right)^{\vee} \hookrightarrow \operatorname{Hom}\left(R^{1/p}_Q \otimes H^i_{\mathfrak{m}}(R_Q), E_{R_Q}(K_Q)\right).$$

We conclude the map

$$R^{1/p}_Q \otimes H^i_{\mathfrak{m}}(R_Q) \twoheadrightarrow H^i_{\mathfrak{m}}\left(R^{1/p}_Q\right)$$

is surjective. ∎

Definition 3.2.21. We say that an R-module M is *F-anti-nilpotent* if the Frobenius action on $H^i_{\mathfrak{m}}(M)$ is anti-nilpotent for every $i \geq 0$.

In particular if $M = R$ we have the following definition.

Definition 3.2.22. We say that a local ring $(R, \mathfrak{m})$ is *stably FH-finite* if $H^i_{\mathfrak{m}}(R)$ is F-anti-nilpotent.

Remark 3.2.23. By Lemma 3.2.7, F-anti-nilpotent implies F-full.

Proposition 3.2.24. *Let $(R, \mathfrak{m})$ be a F-finite and F-pure local ring. Then $H^i_{\mathfrak{m}}(R)$ is stably FH-finite.*

Proof. Let $W \subseteq H^i_{\mathfrak{m}}(R)$ be a F-stable submodule. We want to show that

$$F^e : \frac{H^i_{\mathfrak{m}}(R)}{W} \to \frac{H^i_{\mathfrak{m}}(R)}{W}$$

is injective. Let $y \in H^i_{\mathfrak{m}}(R)$ such that $F^e(y) \in W$. It suffices to show that

$$y \in \left(F(y), F^2(y), \dots\right)$$

since W is F-stable. By Remark 2.2.8, there exists $e' \geq 0$ such that

$$F^{e'}(y) \in \left(F^{e'+1}(y), F^{e'+2}(y), \dots\right) R.$$

Thus for some $n \in \mathbb{N}$ and $r_1, \dots, r_n \in R$ such that

$$F^{e'}(y) = \sum_{i=1}^{n} r_i F^{e'+i}(y).$$

On the other hand, since R is F-finite and F-pure, R is F-split. Hence there exists a splitting map $\varphi : R^{1/p^e} \to R$. Identifying R^{1/p^e} with R, we get that $\varphi \circ F = Id_R$. Since localization and direct limits are exact functors, the map $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ also splits via the map $\bar{\varphi}$ induced via φ. Furthermore, for every $r \in R$ and $\eta \in H^i_{\mathfrak{m}}(R)$ we have that

$$\bar{\varphi}(r\eta) = \varphi(r)\,\bar{\varphi}(\eta).$$

Thus

$$\begin{aligned} y &= Id_R(y) \\ &= \bar{\varphi}^{e'}\left(F^e(y)\right) \\ &= \sum_{i=1}^{n} \bar{\varphi}^{e'}\left(r_i F^{e'+i}(y)\right) \\ &= \sum_{i=1}^{n} \varphi^{e'}(r_i)\, F^i(y) \in W. \end{aligned}$$

We conclude that F^e is injective in $\dfrac{H^i_{\mathfrak{m}}(R)}{W}$. ■

Remark 3.2.25. Cohen-Macaulay rings are F-full. Let R be a Cohen-Macaulay ring and $x_1, \dots, x_d$ be a system of parameters. Similarly to what we did in Example 3.2.9, we have that

$$R^{1/p^e} \otimes H^d_{\mathfrak{m}}(R) = \left[H^d_{\mathfrak{m}}\left(R^{1/p^e}\right)\right].$$

3.3 F-injectivity

In this section we give the definition of an F-injective ring, the definition of finiteness dimension and some relations with depth.

Through this section we assume that R is a local Noetherian ring of characteristic $p > 0$ with maximal ideal $\mathfrak{m}$.

Definition 3.3.1. A ring R is called F*-injective* if the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for every i.

Remark 3.3.2. Note that F-anti-nilpotent implies F-injective, taking the submodule $\{0\}$ of $H^i_{\mathfrak{m}}(R)$.

Whenever the ring is Cohen-Macaulay, we can test F-injectivity in a quotient ring instead of the ring.

Theorem 3.3.3. *Let x a nonzero divisor on R. If R/xR is Cohen-Macaulay and F-injective, then R is Cohen-Macaulay and F-injective.*

Proof. Note that

$$\operatorname{depth} R/xR = \operatorname{depth} R - 1$$

and $\dim R/xR = \dim R - 1$, therefore R is Cohen-Macaulay. Now, to prove that R is F-injective, consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/xR & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/xR & \longrightarrow & 0 \end{array}$$

where the map x is the map sending each element to its multiplication by x. This diagram induces the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R/xR) & \longrightarrow & H^d_{\mathfrak{m}}(R) & \xrightarrow{x} & H^d_{\mathfrak{m}}(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R/xR) & \longrightarrow & H^d_{\mathfrak{m}}(R) & \xrightarrow{x} & H^d_{\mathfrak{m}}(R) & \longrightarrow & 0 \end{array}$$

Suppose the map $x^{p^e-1}F^e$ is not injective. Then there exists $r \in \operatorname{Soc}\left(H^{d-1}_{\mathfrak{m}}(R/xR)\right) \cap \operatorname{Ker}\left(x^{p^e-1}F^e\right)$ nonzero. Note that $\operatorname{Ker}(x) = H^{d-1}_{\mathfrak{m}}(R/xR)$, and so $\operatorname{Soc}\left(H^{d-1}_{\mathfrak{m}}(R/xR)\right) \subseteq H^{d-1}_{\mathfrak{m}}(R/xR)$. Since the diagram commutes

$$F^e(r) = x^{p^e-1}F^e(r) = 0,$$

which is a contradiction for R/xR is F-injective. We conclude $x^{p^e-1}F^e$ is injective and so F^e is also injective on $H^d_{\mathfrak{m}}(R)$. ■

To relate F-injectivity with depth we give the following definition.

Definition 3.3.4. Let M be a finitely generated R-module. The *finiteness dimension* of M with respect to $\mathfrak{m}$ is defined as follows

$$f_{\mathfrak{m}}(M) = \inf\left\{i \mid H^i_{\mathfrak{m}}(M) \text{ is not finitely generated}\right\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

Remark 3.3.5.

1. Assume $\dim M = 0$ or $M = 0$. Then $H^i_{\mathfrak{m}}(M)$ is finitely generated for every $i \geq 0$ and so $f_{\mathfrak{m}} = \infty$. Furthermore, let M be a finitely generated R-module such that $d = \dim M > 0$. Then $H^d_{\mathfrak{m}}(M)$ is not finitely generated, and so $f_{\mathfrak{m}}(M) \leq d$.

2. Suppose R is an image of a Cohen-Macaulay ring. By Grothendieck Finiteness Theorem, we have that
$$f_{\mathfrak{m}}(M) = \min\{\operatorname{depth} M_p + \dim R/p \mid p \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}\}.$$

3. M is generalized Cohen-Macaulay if and only if $\dim M = f_{\mathfrak{m}}(M)$.

4. $\operatorname{depth} R \leq f_{\mathfrak{m}}(R) \leq \dim R$

Theorem 3.3.6 ([MQ18])**.** *Let $x \in R$ be a regular element. If $R/(x)$ is F-injective, then* $\operatorname{depth} R = f_{\mathfrak{m}}(R)$

Proof. Let $t = \operatorname{depth} R$. We proceed by contradiction. Suppose $t < f_{\mathfrak{m}}(R)$. We have the following diagram
$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0
\end{array}$$
which induces
$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{t-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^t_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^t_{\mathfrak{m}}(R) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\
0 & \longrightarrow & H^{t-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^t_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^t_{\mathfrak{m}}(R) & \longrightarrow & \cdots
\end{array}$$
were both α and the left vertical map F are both injective. Since $t < f_{\mathfrak{m}}(R)$, $H^t_{\mathfrak{m}}(R)$ has finite length, and so $m^{p^e-1}H^t_{\mathfrak{m}}(R) = 0$ for some, $e \gg 0$. This is, the map $x^{p^e-1}F^e$ vanishes for some $e \gg 0$. Since x is a regular element, F^e vanishes on $H^t_{\mathfrak{m}}(R)$ for some $e \gg 0$, which is a contradiction. ■

Chapter 4

Filter regular sequence and associated primes

Through this chapter we define filter regular sequences and we relate them to associated primes in the local cohomology groups. In Section 4.1 we define filter regular sequence, and in Section 4.2 we give some results regarding associated primes. This chapter is based on the work of Hailong Dao and Pham Quy [DQ20]. The result about finite associated primes in the local cohomology was also proved by Melvin Hochster and Luis Núñez-Betancourt [HNB17].

4.1 Filter regular sequence

Our goal in this section is to define filter regular sequences and mention some results about them.

In this section R is a Noetherian commutative ring.

Definition 4.1.1. Let $(R, \mathfrak{m}, K)$ be a local ring, M be a finitely generated R-module, and let $x_1, \ldots, x_t \in R$. We say that $x_1, \ldots, x_l$ form a *filter regular sequence on* M if the following conditions hold

- $(x_1, \ldots, x_t) \subseteq \mathfrak{m}$,

- for $i \in \{2, \ldots t\}$, $x_i \notin \mathfrak{P}$ for every $\mathfrak{P} \in \operatorname{Ass}\left(M/\left(x_1, \ldots, x_{i-1}\right) M\right) \setminus \{\mathfrak{m}\}$.

Remark 4.1.2. Using the Prime Avoidance Lemma and that R is a Noetherian ring, we can inductively get a I-filter regular sequence for any $t \geq 1$. First consider the case $t = 1$. By the Prime Avoidance theorem, there exists $x_1 \in I$ such that $x_1 \notin P$ for every $P \in \operatorname{Ass}_R R \setminus \{\mathfrak{m}\}$.

Now for the case $t = 2$, consider the element x_1 we obtained for the previous case. We have that $R/\left(x_1\right)$ is also Noetherian. By the Prime Avoidance Theorem there exists $x_2 \in I$ such that $x_2 \notin P$ for every $P \in \operatorname{Ass}_R \dfrac{R}{(x_1)} \setminus \{\mathfrak{m}\}$. Continuing with this process we can get t elements that satisfy the condition on filter regular sequence.

Now we give some equivalence definition of filter regular sequences.

Lemma 4.1.3. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, K)$. Then $x_1, \ldots, x_t \in \mathfrak{m}$ form a filter regular sequence if and only if one of the following conditions holds:*

- *The quotient*
$$((x_1, \ldots, x_{i-t})\, M :_M x_i) \,/\, (x_1, \ldots, x_{i-t})\, M$$
is an R-module of finite length for every $i = 1, \ldots, t$.

- *Fix $i \in \mathbb{N}$ with $1 \leq i \leq t$. Then the sequence*
$$\frac{x_1}{1}, \ldots, \frac{x_i}{1}$$
forms an $R_\mathfrak{p}$-regular sequence in $M_\mathfrak{p}$ for every $\mathfrak{p} \in (\operatorname{Spec}(R/(x_1 \ldots, x_i)) \cap \operatorname{Supp}_R M) \setminus \{\mathfrak{m}\}$

- *The sequence $x_1^{n_t}, \ldots, x_t^{n_t}$ is a filter regular sequence for all $n_1, \ldots, n_t \geq 1$.*

Finally, we mention the Nagel Schezel Isomorphism, which can be proven using Grothendick spectral sequences.

Lemma 4.1.4 (Nagel-Schenzel Isomorphism)**.** *Let $(R, \mathfrak{m}, K)$ be a local ring and let M be a finitely generated R-module. Let $x_1, \ldots, x_t$ be a filter regular sequence on M. Then we have*
$$H^i_m(M) \cong \begin{cases} H^i_{(x_1,\ldots,x_k)}(M) & \text{if } i < k \\ H^{i-k}_m\left(H^k_{(x_1,\ldots,x_k)}(M)\right) & \text{if } i \geq k \end{cases}$$

4.2 Associated primes

In this section we give some results about associated primes of local cohomology. We aim to show that in finite F-representation type rings the local cohomology groups have a finite number of associated primes.

During this section, R will denote a reduced Noetherian commutative ring with positive characteristic $p > 0$.

The associated primes of a module do not change when we take p roots.

Lemma 4.2.1. *Suppose M is an R-module. Then $\operatorname{Ass}_R M = \operatorname{Ass} M^{1/p^e}$.*

Proof. First note that if $Q = (q_1, \ldots, Q_2) \in \operatorname{Ass}_R(M)$ then by Proposition 2.2.4, $H^0_Q(M_Q) = \Gamma_Q(M_Q)$. Then there exists $l \in M$ such that $ql = 0$ for every $q \in Q$, and so $l \in H^0_Q(M_Q)$. Conversely, if $l \in H^0_Q(M_Q)$, then there exists $n \in \mathbb{N}$ such that $Q^n l = 0$. Since $l \in M_Q$ is an equivalence class, we can take $l \in M$. Therefore $q_1^{n-1} l \in M$ and $Q q_1^{n-1} l = 0$. We have that $Q \in \operatorname{Ass}_R(M)$. Similarly $Q \in \operatorname{Ass}_R\left(M^{1/p^e}\right)$ if and only if $H^0_Q\left(M_Q^{1/p^e}\right) \neq 0$.

Since Frobenius is injective, $H^0_Q(M_Q) \neq 0$ if and only if $\left[H^0_Q(M_Q)\right]^{1/p^e} \neq 0$. Furthermore, this happens if and only if $H^0_Q\left(M_Q^{1/p^e}\right) \neq 0$, because Frobenius commutes with localizations. We conclude $Q \in \operatorname{Ass}_R(M)$ if and only if $Q \in \operatorname{Ass}_R\left(M^{1/p^e}\right)$. ■

Lemma 4.2.2. *Let $I \subseteq R$ be an ideal. Then*
$$\bigcup_{e \geq 0} \operatorname{Ass} \frac{R}{I^{p^e}} \subseteq \operatorname{Ass} \frac{R}{I} \cup \operatorname{Sing} R,$$
where $\operatorname{Sing} R = \{P \in \operatorname{Spec} R \mid R_p \text{ is not regular }\}$.

Proof. Let $P \in \operatorname{Ass} R/I^{[q]}$ for some $q = p^e$. If $P \in \operatorname{Ass} R/I$, then we are done. Otherwise, we show that $P \in \operatorname{Sing} R$. We proceed by contradiction. Suppose R_P is regular. By Kunz Theorem, the Frobenius map is flat over R_p. Since the Frobenius map commutes with localization, we have that

$$\begin{aligned}
\operatorname{depth}\left(R/I^{[q]}\right)_P &= \min\left\{i \mid H^i_{\mathfrak{m}}\left(\left(R/I^{[q]}\right)_P\right) \neq 0\right\} \\
&= \min\left\{i \mid H^i_{\mathfrak{m}}\left(F\left(R/I^{[q]}\right)_P\right) \neq 0\right\} \\
&= \min\left\{i \mid F\left(H^i_{\mathfrak{m}}\left(R/I\right)\right) \neq 0\right\} \\
&= \min\left\{i \mid H^i_{\mathfrak{m}}\left(R/I\right) \neq 0\right\} \\
&= \operatorname{depth}\left(R/I\right)_P .
\end{aligned}$$

Thus, we have $\operatorname{depth}\left(R/I^{[q]}\right)_P = \operatorname{depth}\left(R/I\right)_P > 0$. But this is contradiction since $P \notin \operatorname{Ass} R/I$, and so $(R/I)_P = 0$. ■

Now we prove the main theorem of this chapter.

Theorem 4.2.3 ([DQ20])**.** *Let R have finite F-representation type. Then $H^t_I(R)$ has only finitely many associated primes for any ideal I and any $t \geq 0$.*

Proof. For $t = 0$, we have that

$$H^t_I(R) = \Gamma_I(R) \subseteq R.$$

Since R is Noetherian, $H^t_I(R)$ has finitely many primes.

Now, consider the case $t \geq 1$. By Remark 4.1.2, we can take $x_1, \ldots, x_t$ a filter regular sequence and let $J = (x_1, \ldots, x_t)$. By Nagel-Schenzel Isomorphism, we have that

$$H^t_I(R) \cong H^0_I\left(H^t_J(R)\right) = \Gamma_I\left(H^t_J(R)\right) \subseteq H^t_J(R),$$

hence $\operatorname{Ass}_R H^t_I(R) \subseteq \operatorname{Ass}_R H^t_J(R)$. By Remark 2.2.16 we have that

$$H^t_J(R) \cong \lim_{e\in\mathbb{N}} \frac{R}{\left(x_1^{p^e}, \ldots, x_t^{p^e}\right)},$$

and so

$$\operatorname{Ass}_R H^t_J(R) \subseteq \cup_{e\in\mathbb{N}} \operatorname{Ass}_R \frac{R}{\left(x_1^{p^e}, \ldots, x_t^{p^e}\right)}.$$

Since R has finite F-representation type, there exist finitely generated R-modules $M_1, \ldots, M_s$ and integers $n_{e,1}, \ldots, n_{e,s}$ such that

$$R^{1/p^e} \cong \oplus_{i=1}^s M_i^{n_{e,i}}.$$

Thus

$$\begin{aligned}
\left(R/J^{[q]}\right)^{1/q} &\cong R/J \otimes_R R^{1/q} \\
&\cong R/J \otimes_R \left(\oplus_{i=1}^s M_i^{n_{e,i}}\right) \\
&\cong \oplus_{i=1}^s \left(M_i/JM_i\right)^{n_{e,i}} .
\end{aligned}$$

By Lemma 4.2.1 we have

$$\operatorname{Ass}_R R/J^{[q]} = \operatorname{Ass}_R \left(R/J^{[q]}\right)^{1/q} \subseteq \bigcup_{1=1}^{s} \operatorname{Ass}_R \frac{M_i}{JM_i}.$$

Since each R-module M_i is finitely generated, M_i has finitely many associated primes, for every $i = 1, \ldots, s$. We conclude $H_I^t(R)$ has only finitely many associated primes. ∎

Chapter 5

Frobenius closed parameter ideals

During this chapter we relate Frobenius closure and parameter ideals. We begin in Section 5.1 defining Frobenius closure and giving some results about it. In Section 5.2 we prove when Frobenius closure is equivalent to F-injectivity. This chapter is based on the paper written by Pham Quy and Kazuma Shimomoto [QS17].

A convention for this chapter is that R represents a Noetherian ring of characteristic $p > 0$.

5.1 Frobenius closure

In this section we give the definition of Frobenius closure and some facts about it, specially regarding regular sequences.

Definition 5.1.1. Let $I = (x_1, \ldots, x_l) \subseteq R$ be an ideal. The *Frobenius closure* of I is defined as

$$I^F = \left\{ u \in R \mid u^q \in I^{[q]}, q = p^e \right\}.$$

The Frobenius closure is not only a subset of a ring, it is an ideal.

Remark 5.1.2. Note that

- $0 \in I$, and so $0 \in I^{[q]}$, with $q = p^e$ for every $e \in \mathbb{N}$;
- for $u, v \in I^F$, there exist $q = p^e$ and $q' = p^{e'}$ such that $u^q \in I^{[q]}$ and $v^{q'} \in I^{\left[q'\right]}$. Then we have $u^{q+q'}, v^{q+q'} \in I^{\left[q+q'\right]}$, and so

$$(u + v)^{q+q'} = u^{q+q'} + v^{q+q'} \in I^{\left[q+q'\right]}.$$

- for $r \in R$ and $u \in I^F$, there exists $q = p^e$ such that $ru^q \in I^{[q]}$, and so $(ru)^q \, I^{[q]}$.

We conclude I^F is an ideal of R.

Definition 5.1.3. We say that an ideal $I \subseteq R$ is *Frobenius closed* if $I = I^F$

Frobenius closure implies the ring is reduced under certain conditions.

Proposition 5.1.4. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$. Suppose $x \in R$ is an element such that (x^n) is Frobenius closed for every $n \in \mathbb{N}$. Then R is reduced.*

Proof. Let $u \in R$ be an element such that $u^n = 0$ for some $n \in \mathbb{N}$. Then for every $q = p^e > n$ we have that

$$u^q = 0 \in (x^q).$$

This implies that for every $n \in \mathbb{N}$

$$u \in (x^n)^F = (x^n).$$

By the Krull Intersection Theorem we have that

$$u \in \cap_{n \in \mathbb{N}} (x^n) = 0.$$

We conclude $u = 0$. ■

We go a bit further taking a Frobenius closed ideal generated by regular sequence.

Proposition 5.1.5. *Let $x_1, \ldots, x_l \in R$ be a sequence of elements such that forms a regular sequence in any order and*

$$(x_1, \ldots, x_l) = (x_1, \ldots, x_l)^F.$$

Then for all integers $n_1, \ldots, n_l \in \mathbb{N}$

$$(x_1^{n_1}, \ldots, x_l^{n_l}) = (x_1^{n_1}, \ldots, x_l^{n_l})^F.$$

Proof. First note that for every $a \in (x_1^{n_1}, \ldots, x_l^{n_l})$, $a^{p^0} \in (x_1^{n_1}, \ldots, x_l^{n_l})^{[p^0]}$. Therefore $a \in (x_1^{n_1}, x_2^{n_2}, \ldots, x_l^{n_l})^F$.

Now, we prove the other containment. Since $x_1, \ldots, x_l$ form a regular sequence, it suffices to prove that $(x_1^n, x_2 \ldots, x_l)$ is Frobenius closed. We proceed by induction. The case $n = 1$ is already done by hypothesis. Suppose the conditions holds for $n-1$. Let $a \in (x_1^n, x_2 \ldots, x_l)^F$. We have that for some $q = p^e$

$$\begin{aligned} a^q \in (x_1^n, x_2 \ldots, x_l)^{[q]} \subseteq (x_1, x_2 \ldots, x_l)^{[q]} &\Rightarrow a \in (x_1, x_2 \ldots, x_l)^F = (x_1, x_2 \ldots, x_l) \\ &\Rightarrow a = b_1 x_1 + \cdots + b_l x_l \\ &\Rightarrow a^q = b_1^q x_1^q + \cdots + b_l^q x_l^q \\ &\Rightarrow b_1^q x_1^q \in (x_1^n, x_2, \ldots, x_l)^{[q]} \\ &\Rightarrow b_1^q x_1^q - c x_1^{nq} \in (x_2, \ldots, x_l)^{[q]}, \end{aligned}$$

for some suitable $c \in R$. Recall $x_1, \ldots, x_l$ form a regular sequence in any order, so we can take the element $z = b_1^q - c x_1^{(n_1-1)q} \in (x_2, \ldots, x_l)^{[q]}$. Thus

$$\begin{aligned} b_1^q = z + x_1^{(n_1-1)q} \in \left(x_1^{(n_1-1)}, x_2, \ldots, x_l\right)^{[q]} &\Rightarrow b_1 \in \left(x_1^{(n_1-1)}, x_2, \ldots, x_l\right)^F \\ &\Rightarrow b_1 \left(x_1^{(n_1-1)}, x_2, \ldots, x_l\right) \text{ by induction hypothesis} \\ &\Rightarrow a = b_1 x_1 + \cdots + b_l x_l \in \left(x_1^{(n_1-1)}, x_2, \ldots, x_l\right) \end{aligned}$$

■

Finally, we give a theorem regarding localization in reduced rings.

Theorem 5.1.6. *Frobenius closure commutes with localization. In particular, the localization of a Frobenius closed ideal is Frobenius closed.*

Proof. Let R be a reduced ring, and $I \subseteq R$ an ideal. Now, we have that

$$\begin{aligned} u \in I^F &\Leftrightarrow u^q \in I^{[q]}, \text{ for some } q = p^e \\ &\Leftrightarrow \phi(u) \in IR^\infty \\ &\Leftrightarrow u \in \phi^{-1}(IR^\infty) \cap R. \end{aligned}$$

Therefore $I^F = \phi^{-1}(IR^\infty) \cap R$. We denote $\phi^{-1}(IR^\infty) \cap R$ simply as $IR^\infty \cap R$.

Let $S \subseteq R$ be a multiplicative set. Recall that localization is an exact functor, so it commutes with direct limits. This is,

$$S^{-1}R^\infty \cong \left(S^{-1}R\right)^\infty.$$

Furthermore, we have that $\left(S^{-1}I\right)^F = I\left(S^{-1}R^\infty\right) \cap S^{-1}R$.

Finally, we have that

$$\begin{aligned} S^{-1}\left(I^F\right) &= S^{-1}\left(IR^\infty \cap R\right) \\ &= S^{-1}\left(IR^\infty\right) \cap S^{-1}R \\ &= I\left(S^{-1}R^\infty\right) \cap S^{-1}R \\ &= \left(S^{-1}I\right)^F \end{aligned}$$

■

5.2 Frobenius closed parameter ideals

Parameter ideals and Frobenius closure give us an equivalence to F-injectivity in Cohen-Macaulay rings. Proving this equivalence is the goal in this section. First we prove if a system of parameters is Frobenius closed, then so it a part of it.

Lemma 5.2.1. *If every ideal generated by a full system of parameters is Frobenius closed, then so is every ideal generated by a part of a system of parameters.*

Proof. Let $(x_1, \ldots, x_n)$ be a system of parameters. Without lost of generality take $(x_1, \ldots, x_t)$, part of the system of parameters. Let $y \in (x_1, \ldots, x_t)^F$. Then for every $s \geq 0$

$$y \in (x_1, \ldots, x_t)^F \subseteq \left(x_1, \ldots, x_t, x_{t+1}^s, \ldots, x_n^s\right)^F = \left(x_1, \ldots, x_t, x_{t+1}^s, \ldots, x_n^s\right).$$

Therefore

$$y \in \cap_{s \in \mathbb{N}} \left(x_1, \ldots, x_t, x_{t+1}^s, \ldots, x_n^s\right) = (x_1, \ldots, x_t).$$

Since $(x_1, \ldots, x_t,)^F \subseteq (x_1, \ldots, x_t)$ we are done. ■

Now we prove Frobenius closure implies F-injectivity.

Theorem 5.2.2. *Let $(R, \mathfrak{m}, K)$ be a local ring. Set $d = \dim R$. Then*

1. *Assume* $x_1, \ldots, x_t$ *is a filter regular sequence such that*

$$\left(x_1^{p^n}, \ldots, x_t^{p^n}\right) = \left(x_1^{p^n}, \ldots, x_t^{p^n}\right)^F$$

for all $n \geq 0$*. Then the Frobenius action on* $H^k_{\mathfrak{m}}(R)$ *is injective.*

2. *If every parameter ideal of* R *is Frobenius closed, then* R *is F-injective.*

Proof.

1. Set $I = (x_1, \ldots, x_t)$. By hypothesis $I^{[p^n]}$ is Frobenius closed, thus we have the commutative diagram

$$\begin{array}{ccccccc} R/I & \xrightarrow{\varphi} & R/I^{[p]} & \xrightarrow{\varphi} & R/I^{[p^2]} & \xrightarrow{\varphi} & \ldots \\ \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle \tilde{F}} & & \\ R/I^{[p]} & \xrightarrow{\varphi} & R/I^{[p^2]} & \xrightarrow{\varphi} & R/I^{[p^3]} & \xrightarrow{\varphi} & \ldots \end{array}$$

where φ is the multiplication map by $(x_1, \ldots, x_t)^{p^e - p^{e-1}}$ and $\tilde{F}$ is the map taking each element to its p^e power. Furthermore, note that each $\tilde{F}$ is injective. Taking direct limits we get the Frobenius map

$$F : H^t_I(R) \to H^t_I(R).$$

Furthermore, since $I^{[p^n]}$ is Frobenius closed, R is reduced by Proposition 5.1.4, and so $\tilde{F}$ and F are both injective. It remains to prove that the Frobenius map on $H^t_{\mathfrak{m}}(R)$ is injective. By Lemma 4.1.4, $H^t_{\mathfrak{m}}(R) \cong H^0_{\mathfrak{m}}(H^t_I(R))$. Consider the following commutative diagram induced by the previous diagram

$$\begin{array}{ccccccc} H^0_{\mathfrak{m}}(R/I) & \xrightarrow{\varphi} & H^0_{\mathfrak{m}}\left(R/I^{[p]}\right) & \xrightarrow{\varphi} & H^0_{\mathfrak{m}}\left(R/I^{[p^2]}\right) & \xrightarrow{\varphi} & \ldots \\ \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle \tilde{F}} & & \\ H^0_{\mathfrak{m}}\left(R/I^{[p]}\right) & \xrightarrow{\varphi} & H^0_{\mathfrak{m}}\left(R/I^{[p^2]}\right) & \xrightarrow{\varphi} & H^0_{\mathfrak{m}}\left(R/I^{[p^3]}\right) & \xrightarrow{\varphi} & \ldots \end{array}$$

By Proposition 2.2.4, we have $H^0_{\mathfrak{m}}\left(R/I^{[p^n]}\right) \subseteq R/I^{[p^n]}$ for every $n \in \mathbb{N}$. Therefore, we have the following commutative diagram

$$\begin{array}{ccc} R/I^{[p^n]} & \overset{\tilde{F}}{\hookrightarrow} & R/I^{[p^{n+1}]} \\ \uparrow & & \uparrow \\ H^0_{\mathfrak{m}}\left(R/I^{[p^n]}\right) & \xrightarrow{\tilde{F}} & H^0_{\mathfrak{m}}\left(R/I^{[p^{n+1}]}\right), \end{array}$$

which implies that every $\tilde{F}$ is injective. Note that taking direct limit of both lines we get the Frobenius map on $H^t_{\mathfrak{m}}(R)$. Since local cohomology commutes with direct limits and $\tilde{F}$ is injective, we conclude that R is F injective.

2. By the Prime Avoidance Theorem we can get a filter regular sequence. Furthermore, since the set of minimal primes are contained in the set of associated primes, this filter regular sequence is also a system of parameters. Let $\{x_1, \ldots, x_d\}$ be such sequence and $I = (x_1, \ldots, x_t)$ with $0 \leq t \leq d$. By hipothesis I is Frobenius closed, and so by Lemma 5.2.1, $I^{[p^n]}$ is Frobenius closed for every $n \in \mathbb{N}$. Finally, by 1, we conclude R is F-injective.

■

Finally, we prove the equivalence we mention in the beginning of this section.

Corollary 5.2.3 ([QS17]). *Let $(R, \mathfrak{m}, K)$ be a local Cohen-Macaulay ring. The following are equivalent*

1. *every parameter ideal is Frobenius closed;*
2. *there is a parameter ideal of R that is Frobenius closed;*
3. *R is F-injective.*

Proof. Suppose there exists a parameter ideal $I = (x_1, \ldots, x_k)$ which is Frobenius closed. Since R is Cohen-Macaulay, $\{x_1, \ldots, x_k\}$ is also a regular sequence. By Proposition 5.1.5 $(x_1^{n_1}, \ldots, x_k^{n_k})$ is Frobenius closed for all integers $n_1, \ldots, n_k$. Finally, Theorem 5.2.2, implies that R is F-injective.

Finally, suppose R is F-injective. Let $x_1, \ldots, x_d$ be a system of parameters. Since R is Cohen-Macaulay, $x_1, \ldots, x_d$ is also a regular sequence. Let $c = \left[\frac{a}{x_1^{t_1} \cdots x_d}\right] \in H_{\mathfrak{m}}^d(R)$ for some $\frac{a}{x_1^t \cdots x_d^t} \in C^d$ (see Remark 2.2.12). Note that $c = 0$, then $\frac{a}{x_1^t \cdots x_d^t} \in \sum_{i=1}^d R_{x_1 \cdots \hat{x_i} \cdots x_d}$. Thus we have that for some $c_1, \ldots, c_d \in R$

$$\frac{a}{x_1^t \cdots x_d^t} = \sum_{i=1}^d \frac{c_i}{x_1^t \cdots x_d^t}$$
$$\Leftrightarrow a = \sum_{i=1}^d c_i x_i^t$$
$$\Leftrightarrow a \in \left(x_1^t, \ldots, x_d^t\right)$$

Now if $u \in (x_1, \ldots, x_d)^F$, there exists $q = p^e$ for some $e > 0$ such that $u^q \in (x_1, \ldots, x_d)^{[q]}$. Therefore

$$\left[\frac{u}{x_1 \cdots x_d}\right]^q = \left[\frac{u^q}{x_1^q \cdots x_d^q}\right] = 0$$

Since R is F-injective, we have $\left[\frac{u}{x_1 \cdots x_d}\right] = 0$ and so $u \in (x_1, \ldots, x_d)$.

Finally, recall we can always get a filter regular sequence, which is a system of parameters. Hence we have the equivalence from 1 to 2. ■

Chapter 6

F-purity

This chapter is about F-purity. We begin Section 6.1 talking about purity in order to prove the equivalence between F-purity and F-splitness. We also give conditions for the equivalence between F-injectiveness and F-purity and characterized F-purity using the Frobenius closure. We finish the section proving that regularity implies F-purity. In Section 6.2 we give some properties of anti-nilpotent modules which are applied in the following section. In Section 6.3 we review an example of an F-injective ring that is not F-pure. This example was given by Pham Quy and Kazuma Shimomoto [QS17].

6.1 Purity

In this section we first recall some propositions about pure maps in rings not necessarily of charateristic $p > 0$. Hence, just for this section, R denotes a Noetherian commutative ring of not necessarily prime characteristic. If M is an R-module, then assume M is finitely generated. Later in this section we work with prime characteristic rings in order to give an equivalent definition of F-purity and an equivalence with F-injectivity. Then we prove F-purity implies Frobenius closed.

We recall a result from pure maps

Lemma 6.1.1. *Let $(R, \mathfrak{m})$ be a local ring. Then the map $R \to M$ is pure if and only if $E \otimes_R R \to E \otimes_R M$ is injective, where E is the injective hull of $R/\mathfrak{m}$.*

We want to prove the equivalence between purity and splitness. First we see splitness implies purity.

Proposition 6.1.2. *Let R be a domain and $\varphi : M \to N$ an R-linear map that splits. Then, φ is pure.*

Proof. As φ splits, we have that $N = M \oplus S$ for some R-module S. Consider an R-module T. Then

$$N \otimes_R T = (M \otimes_R T) \oplus (S \otimes_R T)\,.$$

Hence the map $\varphi \otimes_R Id_T : M \otimes_R T \to N \otimes_R T$ is injective. We conclude φ is pure. ∎

To see that purity implies splitness we need some results

Theorem 6.1.3. *Let R be a ring,*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence, and

$$\varphi : F_1 \longrightarrow F_0$$

be a homomorphism of free modules of finite rank. Let $M = \operatorname{Coker}\varphi$ *and* $M' = \operatorname{Coker}\varphi^*$*. Then*

$$\operatorname{Ker}(M' \otimes E' \to M' \otimes E) \cong \operatorname{Coker}(\operatorname{Hom}(M, E) \to \operatorname{Hom}(M, E'')).$$

Proof. Consider

$$E : 0 \longrightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \longrightarrow 0$$

and

$$F : \qquad F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\pi} M \longrightarrow 0.$$

We have the following sequences

$$0 \longrightarrow E''^* \xrightarrow{\beta^*} E^* \xrightarrow{\alpha^*} E'^*$$

and

$$0 \longrightarrow M^* \xrightarrow{\pi^*} F_0^* \xrightarrow{\varphi^*} F_1^* \xrightarrow{\theta} M' \longrightarrow 0.$$

Form the following double complex

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
M^* \otimes E' & \longrightarrow & M^* \otimes E & \longrightarrow & M^* \otimes E'' \\
\downarrow & & \downarrow & & \downarrow \\
F_0^* \otimes E' & \longrightarrow & F_0^* \otimes E & \longrightarrow & F_0^* \otimes E'' \\
\downarrow & & \downarrow & & \downarrow \\
F_1^* \otimes E' & \longrightarrow & F_1^* \otimes E & \longrightarrow & F_1^* \otimes E''.
\end{array}$$

The modules F_i are free, so they are projective. In addition, $\operatorname{Hom}(F_i, \bullet) \cong \operatorname{Hom}(F_i, R) \otimes \bullet$, so we get

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^* \otimes E' & \longrightarrow & M^* \otimes E & \longrightarrow & M^* \otimes E'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0^* \otimes E' & \longrightarrow & F_0^* \otimes E & \longrightarrow & F_0^* \otimes E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1^* \otimes E' & \longrightarrow & F_1^* \otimes E & \longrightarrow & F_1^* \otimes E'' & \longrightarrow & 0.
\end{array}$$

By the right exactness of the tensor product, we have the sequence

$$F_0^* \otimes G \xrightarrow{\varphi^* \otimes \mathrm{Id}_G} F_1^* \otimes G \xrightarrow{\theta \otimes \mathrm{Id}_G} M' \otimes G \longrightarrow 0.$$

for every R-module G. Then, $M' \otimes G \cong \mathrm{Coker}\,(\varphi^* \otimes G)$. Now we complete the complex and simplify the notation

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^* \otimes E' & \longrightarrow & M^* \otimes E & \longrightarrow & M^* \otimes E'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0^* \otimes E' & \longrightarrow & F_0^* \otimes E & \longrightarrow & F_0^* \otimes E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1^* \otimes E' & \longrightarrow & F_1^* \otimes E & \longrightarrow & F_1^* \otimes E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' \otimes E' & \longrightarrow & M' \otimes E & \longrightarrow & M' \otimes E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

Applying the Snake's Lemma, we have the exact sequence

$$M^* \otimes E' \longrightarrow M^* \otimes E \longrightarrow M^* \otimes E'' \xrightarrow{d} M' \otimes E' \longrightarrow M' \otimes E \longrightarrow M' \otimes E''.$$

We have that

$$\frac{M^* \otimes E''}{\mathrm{Ker}\,(d)} \cong \mathrm{Im}\,(d)$$

Note that, $\mathrm{Ker}\,(d) = \mathrm{Im}\,(\mathrm{Id}_{M^*} \otimes \beta)$ and $\mathrm{Im}\,(d) = \mathrm{Ker}\,(\mathrm{Id}_{M'} \otimes \alpha)$. Therefore,

$$\begin{aligned}
\mathrm{Coker}\,(\mathrm{Id}_{M^*} \otimes \beta) &= \frac{M^* \otimes E''}{\mathrm{Im}\,(\mathrm{Id}_{M^*} \otimes \beta)} \\
&= \frac{M^* \otimes E''}{\mathrm{Ker}\,(d)} \\
&= \mathrm{Im}\,(d) \\
&= \mathrm{Ker}\,(\mathrm{Id}_{M'} \otimes \alpha)\,.
\end{aligned}$$

■

Corollary 6.1.4. *Let R be a Noetherian ring. Then the exact sequence*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is pure if and only if for every finitely generated module N, the morphism

$$\theta : \mathrm{Hom}\,(N, E) \to \mathrm{Hom}\,(N, E'')$$

is surjective.

Proof. For this part, we use a finitely generated module N. Then, we have an exact sequence

$$0 \longrightarrow K \xrightarrow{w} F \xrightarrow{\varphi} N \longrightarrow 0$$

where both K and F are free modules of finite rank. Consider the R-module map

$$w^* : F^* \to K^*.$$

Note that F^* and K^* are finitely generated. Hence, $\operatorname{Im}(w^*)$ is finitely generated. Let $M = \operatorname{Coker}(w^*)$, which is finitely presented.

Now, suppose the exact sequence from the statement is pure. By Theorem 6.1.3, we have the exact sequence

$$\operatorname{Hom}(N, E) \xrightarrow{j} \operatorname{Hom}(N, E'') \xrightarrow{d} M \otimes E' \xrightarrow{h} M \otimes E.$$

Since h is injective, we get $\operatorname{Im} d = \operatorname{Ker} d = 0$. We conclude that j is surjective.

For the converse, we use that M is finitely generated. Note that $w^{**} = w$ and $\operatorname{Coker} w^{**} = N$. Likewise, applying Theorem 6.1.3, we get the exact sequence

$$\operatorname{Hom}(M, E) \longrightarrow \operatorname{Hom}(M, E'') \xrightarrow{\tilde{d}} N \otimes E' \xrightarrow{\tilde{h}} N \otimes E.$$

Then $\operatorname{Ker} \tilde{h} = \operatorname{Im} \tilde{d} = 0$. As the functor $\bullet \otimes N$ is right-exact, we conclude that the exact sequence in the statement is pure.

■

As a corollary we get the follwing.

Corollary 6.1.5 ([HR76]). *Let R be a Noetherian ring. Then the exact sequence*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0.$$

If E'' is finitely generated, then the exact sequence is pure if and only if it splits.

Proof. Let $\theta : E \to E''$ be the morphism in the statement. Suppose the sequence is pure. By the first part, we have

$$\operatorname{Hom}(E'', E) \longrightarrow \operatorname{Hom}(E'', E'') \longrightarrow 0.$$

In particular, there exists a $\varphi \in \operatorname{Hom}(E'', E)$ such that it is the preimage of the identity homomorphism. Therefore

$$(\varphi \circ \theta)(1) = 1.$$

Finally, let N be a finitely generated module, and be φ a splitting for θ. We show that for every $\beta \in \operatorname{Hom}(N, E'')$, there exists an element in $\alpha \in \operatorname{Hom}(N, E)$ such that $\theta \circ \alpha = \beta$. Take $\alpha = \varphi \circ \beta$. Then

$$\begin{aligned}(\theta \circ \alpha)(x) &= (\theta \circ \varphi \circ \beta)(x) \\ &= \beta(x).\end{aligned}$$

By Corollary 6.1.4, we are done. ■

Finally, we have the equivalence of F-purity and F-splitness.

Corollary 6.1.6. *Let R be a Noetherian ring of characteristic p. Let R be F-finite. Then R is F-split if and only if R is F-pure.*

Proof. This follows directly from 6.1.5. ∎

Before proving the equivalence condition to F-purity, we give some results on Gorenstein rings

Lemma 6.1.7. *Let $(S, \mathfrak{m}) \subseteq (S^*, \mathfrak{n})$ be Gorenstein rings, and assume that S^* is a finitely generated free S-module. The*

1. $\operatorname{Hom}_S(S^*, S) \cong S^*$ *as an S^*-module.*
2. *Let T be a generator for* $\operatorname{Hom}_S(S^*, S)$ *as an S^*-module, H be an ideal in S^*, I be an ideal in S and $s \in S^*$. Then the image of H under the isomorphism $sT : S^* \to S$ is contained in I if and only if $s \in (IS^* :_{S^*} H)$.*

Proof. First we prove the isomorphism between $\operatorname{Hom}_S(S^*, S)$ and S^*. Since S and S^* are Gorenstein, Remark 2.2.20 implies that $\omega_S \cong S$ and $\omega_{S^*} \cong S^*$, where ω_S and ω_{S^*} are canonical modules for S and S^*, respectively. Then by Theorem 2.2.21 we have that

$$\begin{aligned} \operatorname{Hom}_S(S^*, S) &\cong \operatorname{Hom}_S(S^*, \omega_S) \\ &\cong \omega_{S^*} \\ &\cong S^*. \end{aligned}$$

Now, we prove 2. Note that $sT(H) \subseteq I$ if and only if $sT(\sigma S^*)$, for every $\sigma \in H$, and this happens if and only if $s\sigma T(S^*)$, for every $\sigma \in H$. Hence, it suffices to prove that $s\sigma \in IS*$, for every $\sigma \in H$.

Let $\{m_1, \ldots, m_n\}$ be a basis for S^* as S-module. Consider the maps $\hat{m_i} \in \operatorname{Hom}_S(S^*, S)$ given by

$$\hat{m_i}(y) = \begin{cases} 1 & y = m_i \\ 0 & y \neq m_i, \end{cases}$$

for every $i \in \{1, \ldots, n\}$. Hence we have a dual basis for given by $\{\phi_i \mid \phi_i = \hat{m_i}T, i = 1, \ldots, n\}$. Therefore we have that for every $\sigma \in H$, $s\sigma T(S^*) \subseteq I$ if and only if for every $i = 1, \ldots, n$, $s\sigma T(m_i) = l_i \in I$. By the choice of basis this happens if and only if $s\sigma T = (\sum_{i=1}^n l_i \hat{m_i})T$. Finally, since T is a generator for $\operatorname{Hom}_S(S^*, S)$ as an S^*-module, this happens if and only if $s\sigma = \sum_{i=1}^n l_i m_i \in IS^*$. ∎

Under the assumptions of Lemma 6.1.7, we have the following corollary.

Corollary 6.1.8. *Let $(S, \mathfrak{m}) \subseteq (S^*, \mathfrak{n})$ be Gorenstein rings, and assume that S^* is a finitely generated free S-module. Let T be a generator for* $\operatorname{Hom}_S(S^*, S)$ *as an S^*-module, H be an ideal in S^*, I be an ideal in S. Then there exists an isomorphism given by*

$$\begin{aligned} \psi : \frac{(IS^* :_{S^*} H)}{IS^*} &\to \operatorname{Hom}_S(S^*/H, S/I) \\ \overline{s} &\mapsto \overline{sT}\left(\overline{t}\right) = \overline{sT(t)}. \end{aligned}$$

Proof. Let $s \in S^*$. By Lemma 6.1.7, we have that $sT(H) \subseteq I$ if and only if $s \in (IS^* :_{S^*} H)$. Furthermore, we have that $s \in IS^*$ implies $sT(S^*) \subseteq I$, and so $\overline{sT} \equiv 0$. Therefore ψ is a well defined function. Since T is a generator, we have that ψ is a homomorphism of S^*-modules.

To see ψ is injective, let $\overline{s} \in \ker \psi$. Then $\overline{sT} = 0$, this is $sT(S^*) \subseteq I$. By Lemma 6.1.7, $s \in IS^*$.

Finally, we see ψ is surjective. Since S^* is a finitely generated free S-module, it is projective. Hence for every $\phi : S^*/H \to S/I$, there exists a map $\phi_0 : S^* \to S$ such that the following diagram commutes

$$\begin{array}{ccc} S^* & \xrightarrow{\pi} & S^*/H \\ {\scriptstyle \phi_0}\downarrow & & \downarrow{\scriptstyle \phi} \\ S & \xrightarrow[\pi]{} & S/I \end{array}$$

Since $\phi_0 \in \operatorname{Hom}_S(S^*, S)$ and T is a generator, there exists $s \in S^*$ such that $\phi_0 = sT$. Hence $\phi = \overline{sT}$. ■

Now we assume that S is a ring of characteristic $p > 0$. Note that $S \cong S^{1/p}$ as rings. Then we have that $IS^{1/p} \cong I^{[p]}$. Then we have the following corollary.

Corollary 6.1.9. *Let S is an F-finite regular local ring and $R = S/I$, where $I \subseteq S$ is an ideal. Let T be a generator of $\operatorname{Hom}_S\left(S^{1/p}, S\right)$ as $S^{1/p}$-module. Then there exists an isomorphism $\psi : \left[\left(I^{[p]} : I\right)/I\right] R^{1/p} \to \operatorname{Hom}\left(R^{1/p}, R\right)$*

$$\psi : \frac{\left(I^{[p]} : I\right)}{I} R^{1/p} \to \operatorname{Hom}_R\left(R^{1/p}, R\right)$$
$$\overline{s} \mapsto \overline{sT}\left(\bar{t}\right) = \overline{sT(t)}.$$

Proof. This follows from Corollary 6.1.8. ■

Theorem 6.1.10 ([Fed83])**.** *Let $(S, \mathfrak{m})$ be a F-finite regular local ting and let $R = S/I$. Then R is F-pure if and only if $\left(I^{[p]} : I\right) \not\subseteq \mathfrak{m}^{[p]}$.*

Proof. By Definition 3.2.15, the Frobenius map on R splits if and only if there exists $\phi \in \operatorname{Hom}_R\left(R^{1/p}, R\right)$, such that $\pi\left(1^{1/p}\right) = 1$, this is, $\phi(R) \not\subseteq \tilde{\mathfrak{m}}$.

Since S is regular, it is Gorenstein. By Lemma 6.1.7, we take T a generator of $\operatorname{Hom}_S\left(S^{1/p}, S\right)$ as $S^{1/p}$-module. Hence $\pi = \overline{sT}$ for some $s \in S^{1/p}$. By Lemma 6.1.7, $sT\left(S^{1/p}\right) \not\subseteq \mathfrak{m}$ if and only if $s \notin \left(\mathfrak{m}^{[p]} : S^{1/p}\right) = \mathfrak{m}^{[p]}$.

■

Under certain conditions, F-injectivity is equivalent to F-purity.

Theorem 6.1.11 ([Fed83])**.** *Let $(R, \mathfrak{m})$ is a local reduced ring of characteristic $p > 0$ of dimension d, and F-finite. If R is F-pure, then R is F-injective. Conversely, if R is Gorenstein, F-finite and F-injective, R is F-pure.*

Proof. First suppose R is F-pure. By Corollary 6.1.5, R is F-split. Using the definition of local cohomology taken from the Koszul cohomology, we get that $H^i_{\mathfrak{m}}(R)$ is F-pure for every i. By Definition 3.2.15,

$$H^i_{\mathfrak{m}}\left(R^{1/p}\right) \cong H^i_{\mathfrak{m}}(R) \oplus H^i_{\mathfrak{m}}(M) \text{ for all } i.$$

Hence R is F-injective.

Now, suppose R is F-injective and Gorenstein. Then we have the injective map

$$H^i_{\mathfrak{m}}(R) \overset{F}{\hookrightarrow} H^i_{\mathfrak{m}}\left(R^{1/p}\right),$$

and so, using the Matlis duality we have the surjective map

$$\left(H^i_{\mathfrak{m}}\left(R^{1/p}\right)\right)^\vee \overset{\alpha}{\twoheadrightarrow} \left(H^i_{\mathfrak{m}}(R)\right)^\vee.$$

Note that

$$\begin{aligned}\left(H^d_{\mathfrak{m}}(R)\right)^\vee &= \operatorname{Hom}_R\left(H^d_{\mathfrak{m}}(R), \mathrm{E_R}(\mathrm{K})\right)\\ &\cong \operatorname{Hom}_R\left(\operatorname{Hom}_R\left(\omega_R, \mathrm{E_R}(\mathrm{K})\right), \mathrm{E_R}(\mathrm{K})\right)\\ &\cong \operatorname{Hom}_R\left(\operatorname{Hom}_R\left(R, \mathrm{E_R}(\mathrm{K})\right), \mathrm{E_R}(\mathrm{K})\right)\\ &\cong R^{\vee\vee}\\ &\cong R.\end{aligned}$$

On the other hand Since $R^{1/p}$ is Gorenstein,

$$\begin{aligned}\left(H^d_{\mathfrak{m}}\left(R^{1/p}\right)\right)^\vee &= \left(R^{1/p} \otimes H^d_{\mathfrak{m}}(R)\right)\vee\\ &\cong \left(R^{1/p} \otimes E_R\right)^\vee\\ &\cong \operatorname{Hom}_R\left(R^{1/p} \otimes E_R, E_R\right)\\ &\cong \operatorname{Hom}_R\left(R^{1/p}, E^\vee\right)\\ &\cong \operatorname{Hom}_R\left(R^{1/p}, R\right)\\ &\cong R^{1/p}.\end{aligned}$$

Thus, there exists $f^{1/p} \in R^{1/p}$ such that $\alpha\left(f^{1/p}\right) = 1$, and so the map $\alpha \circ f^{1/p}$ sends $1^{1/p}$ to 1. We conclude R is F-split, hence F-pure. ■

Now we prove that F-purity implies Frobenius closed.

Theorem 6.1.12. *Let R be an F-pure Noetherian ring of characteristic $p > 0$. Then for every ideal $I \subseteq R$, $I = I^F$.*

Proof. We know that $I \subseteq I^F$. We want to show $I^F \subseteq I$. Let $u \in I^F$. Then there exists $q = p^e$ such that $u^q \in I^{[q]}$ Consider the map

$$R \overset{F}{\longrightarrow} R^{1/q}.$$

Since R is F-pure, the map $F \otimes_R Id_{R/I}$ is injective. Since $R \otimes_R R/I \cong R/I$ and $R^{1/q} \otimes_R R/I \cong R^{1/q}/IR^{1/q}$, the map $F \otimes_R Id_{R/I}$ induces a map

$$R/I \overset{\varphi}{\longrightarrow} R^{1/q}/IR^{1/q}.$$

Note that φ is injective since the following diagram commutes

$$\begin{array}{ccc} R \otimes_R R/I & \hookrightarrow & R^q \otimes_R R/I \\ \cong \downarrow & & \downarrow \cong \\ R/I & \xrightarrow{\varphi} & R^{1/q}/IR^{1/q} \end{array}$$

Recall that $IR^{1/q} = \left(I^{[q]}\right)^{1/q}$. Hence, we have that

$$\varphi\left(\overline{u}\right) = \left[\left(u^q\right)^{1/q}\right] \in IR^q.$$

Since φ is injective, $u \in I$.

■

6.2 F-anti-nilpotent rings

During this section R denotes a Noetherian ring of characteristic $p > 0$. Now we see anti-nilpotent modules in exact sequences.

Lemma 6.2.1.

1. *Let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence of $R\{F\}$-modules. Then M is anti-nilpotent if and only if so are L and N.

2. *Let*

$$L \xrightarrow{\beta} M \xrightarrow{\alpha} N$$

be an exact sequence of $R\{F\}$-modules such that L is anti-nilpotent and F acts injectively on N. Then F acts injectively on M.

Proof. We prove 1. If L and N are both anti-nilpotent, by the Five Lemma M is anti-nilpotent too. Now, suppose M is anti-nilpotent. Let $T \subseteq L$ be an F-stable submodule. Then $T \subseteq M$ is an F-stable submodule of M, hence F acts injectively on M/T. Since the Frobenius action on L/T is the composition of the Frobenius action on M/T and the inclusion $L/T \hookrightarrow M/T$, we have that L is anti-nilpotent.

To prove that N is F-anti-nilpotent, recall that $N \cong M/L$. Hence, is $A \subseteq N$ is an F-stable submodule, we have that $N/A \cong M/\pi^{-1}(A)$, where $\pi : M \to M/L$ is the projection map. Since A is an F-stable N-submodule, $\pi^{-1}(A)$ is an F-stable M-module. We conclude N is F-anti-nilpotent

Now we prove 2. Consider the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\alpha) \longrightarrow M \longrightarrow \operatorname{Im}(\alpha) \longrightarrow 0.$$

We have that $\ker\alpha = \operatorname{Im}\beta \cong L/\operatorname{Ker}\beta$. Since $\operatorname{Ker}\beta \subseteq L$ is a F-stable $R\{F\}$-subquotient and L is anti-nilpotent, F acts injectively on $\operatorname{Ker}\alpha$. On the other hand $\operatorname{Im}\alpha \subseteq N$ is a $R\{F\}$-submodule and F acts injectively on N, hence F acts injectively on $\operatorname{Im}\alpha$. By 1, F acts injectively on M. ■

Theorem 6.2.2 ([QS17]). *Let $(R, \mathfrak{m})$ be a F-finite local ring. Suppose there exist ideals $I, I \subseteq R$ such that $R/(I+J)$ is F-pure, and $R/I, R/J$ are F-injective. Then $R/(I \cap J)$ is F-injective.*

Proof. We have the short exact sequence

$$0 \longrightarrow R/(I \cap J) \xrightarrow{f} R/I \oplus R/J \longrightarrow R/(I+J) \longrightarrow 0$$

were $f(a) = (a, -a)$ and $g(c, b) = c + b$. Note that both f, g commutes with the Frobenius map. This induces the long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{m}}^{i-1}(R/(I+J)) \xrightarrow{h_1} H_{\mathfrak{m}}^{i}(R/(I \cap J)) \xrightarrow{h_2} H_{\mathfrak{m}}^{i}(R/I) \oplus H_{\mathfrak{m}}^{i}(R/J) \longrightarrow \cdots$$

Since $R/(I+J)$ is F-pure, by Proposition 3.2.24, it is F-anti-nilpotent. Note that F acts injectively on $H_{\mathfrak{m}}^{i-1}(R/(I+J))$. Using that both R/I and R/J are F-injective, its direct sum is also F-injective. By Lemma 6.2.1, $R/(I \cap J)$ is F-injective. ∎

Similarly we have the version of this proposition with the property of being F-anti-nilpotent.

Proposition 6.2.3. *Let $(R, \mathfrak{m})$ be a local ring. Suppose there exists ideals $I, J \subseteq R$ such that $R/(I+J)$, R/I and R/J are F-anti-nilpotent. Then $R/(I \cap J)$ is F-anti-nilpotent.*

Proof. We have the short exact sequence

$$0 \longrightarrow R/(I \cap J) \xrightarrow{f} R/I \oplus R/J \longrightarrow R/(I+J) \longrightarrow 0$$

as in Theorem 6.2.2, which induces the long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{m}}^{i-1}(R/(I+J)) \xrightarrow{h_1} H_{\mathfrak{m}}^{i}(R/(I \cap J)) \xrightarrow{h_2} H_{\mathfrak{m}}^{i}(R/I) \oplus H_{\mathfrak{m}}^{i}(R/J) \longrightarrow \cdots$$

Since both $R/I, R/J$ are F-anti-nilpotent, then so is $R/I \oplus R/J$. By Lemma 6.2.1 we have that $R/(I \cap J)$ is F-anti-nilpotent. ∎

6.3 Examples

The goal in this section is to give an example of a ring that is F-injective but not F-pure. During this section K denotes a perfect field of prime characteristic p.

Lemma 6.3.1. *Let $R = K[\![U, V, Y, Z]\!]/\left(UV, UZ, Z\left(V - Y^2\right)\right)$. Then R is F-anti-nilpotent, and so F-injective.*

Proof. First note that

$$\left(UV, UZ, Z\left(V-Y^2\right)\right) = \left(U, V-Y^2\right) \cap (Z,U) \cap (Z,V) = \left(U, V-Y^2\right) \cap (Z, UV)$$

Let $I = \left(U, V-Y^2\right)$ and $J = (Z, UV)$. Then we have that $I+J = \left(U, V-Y^2, Z\right)$. Let $S = K[\![U,V,Y,Z]\!]$. Then we have that both S/I and $S/(I+J)$ are regular rings. By Theorem 3.2.14, both rings are F-pure. Furthermore, since $r = Z^{p-1}U^{p-1}V^{p-1} \in \left((Z,UV)^{[p]} : (Z,UV)\right)$ and $Z \notin (U,V,Y,Z)^{[p]}$, by Fedder's criterion S/J is F-pure. By Proposition 3.2.24 S/J is F-anti-nilpotent. By Proposition 6.2.3, the ring $R/(I\cap J)$ is F-anti-nilpotent. ■

Now we present an example of a F-injective ring which is not F-pure.

Example 6.3.2 ([QS17])**.** Let $S = K[\![U,V,Y,Z,T]\!], I = (T), J = \left(UV, UZ, Z\left(V-Y^2\right)\right), R = S/(I\cap J)$. By Lemma 6.3.1, S/J is F-anti-nilpotent. Since S/I is regular it is F-pure, and so F-anti-nilpotent. Moreover, $S/(I+J) \cong K[\![U,V,Y,Z]\!]/I$, so it is F-anti-nilpotent. Hence, by Theorem 6.2.3, R is also F-anti-nilpotent, and so F-injective.

Now, let u,v,y,z,t be the image of U,V,Y,Z,T in R, respectively. Note that $(t) \in \operatorname{Ass} R$ and

$$R/(t) \cong K[\![U,V,Y,Z]\!]/(t) \cong K[\![U,V,Y,Z]\!].$$

Therefore $\dim R/(t) = 4$. Using Macaulay2, we have that the associated primes of R are $(t), (z,v), (z,u), \left(u, y^2+v\right)$. Furthermore, (t) is the only associated prime satisfying this.

Let $a = y^2\left(u^2 - z^4\right)$. Then a is a parameter element. Since $a \in (u,z)$, then a is a zero divisor in R. We want to show that $(a) \subseteq R$ is not Frobenius closed. Note that in $R/(a)^{[p]}$ we have that

$$\begin{aligned}\left(y^3z^4t\right)^p &= y^{3p}z^{4p}t^p \\ &= y^{3p-2}t^p\left(y^2z^{4p}\right) \\ &= y^{3p-2}t^p\left(vu^{2p}\right) \\ &= y^{3p-2}u^{2p}vt^p \\ &= 0.\end{aligned}$$

Hence $\left(y^3z^4t\right)^p \in (a)^{[p]}$. However, if $y^3z^4t \in a$, then

$$Y^3Z^4T = A\left(Y^2\left(U^2 - Z^4\right)\right) + B(TUV) + C(TUZ) + D\left(TZ\left(V-Y^2\right)\right),$$

for some $A,B,C,D \in S$. Hence $A = TA'$, for some $A' \in S$, and so

$$Y^3Z^4 = A'\left(Y^2\left(U^2-Z^4\right)\right) + B(UV) + C(UZ) + D\left(Z\left(V-Y^2\right)\right).$$

In $S/(U,V)$, we have that

$$Y^3Z^4 = A'\left(-Y^2Z^4\right) + D'\left(ZY^2\right).$$

Furthermore, $A' = -Y$ and $D = D'U$ o $D = D'V$, for some $D' \in S$. Then in S we have that

$$Y^3Z^4 = -Y^3U^2 + Y^3Z^4 + B(UV) + C(UZ) + D\left(Z\left(V-Y^2\right)\right),$$

and so

$$Y^3U^2 = B(UV) + C(UZ) + D\left(Z\left(V-Y^2\right)\right).$$

Thus $D = UD'$, and

$$Y^3U = B(V) + C(Z) + D'\left(Z\left(V - Y^2\right)\right) \in (V, Z),$$

which is a contradiction. We conclude (a) is not Frobenius closure, and so R is not F-pure by Theorem 6.1.12.

Chapter 7

Deformation of singularities

During this chapter we talk about deformation of singularities. In Section 7.1 we define surjective elements and give some properties on local cohomology. Section 7.2 is devoted to prove the deformation of F-full rings and F-anti-nilpotent rings. Finally in Section 7.3 we define strongly F-injectivity and prove the deformation of F-injectivity under certain conditions.

In this chapter we consider a Noetherian local ring $(R, \mathfrak{m})$ of characteristic $p > 0$.

7.1 Surjective elements

During this section we define surjective elements and state relations with some F-singularities. In addition, using these elements, we construct a family of non F-full local rings. Finally we prove F-full implies F-injective.

Definition 7.1.1. Let $x \in R$ be a regular element. We say x is a *surjective element* if the natural map on the local cohomology

$$H^i_{\mathfrak{m}}\left(R/\left(x^n\right)\right) \to H^i_{\mathfrak{m}}\left(R/\left(x\right)\right)$$

induced by $R/\left(x^n\right) \to R/\left(x\right)$ is surjective for every $n > 0$ and for every $i \geq 0$.

Proposition 7.1.2. *The following are equivalent*

1. *x is a surjective element*
2. *For all $0 < h \leq k$ the multiplication map*

$$R/\left(x^h\right) \xrightarrow{x^{k-h}} R/\left(x^k\right)$$

induces an injection

$$H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^k\right)\right)$$

for each $i \geq 0$.

3. *For all* $0 < h \leq k$ *the short exact sequence*

$$0 \longrightarrow R/\left(x^h\right) \xrightarrow{x^{k-h}} R/\left(x^k\right) \longrightarrow R/\left(x^{k-h}\right) \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^k\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^{k-h}\right)\right) \longrightarrow 0$$

for each $i \geq 0$.

Proof. First suppose x is a surjective element. Then the map x^{k-h} is injective. We proceed by induction on n, where $k = l + n$. Suppose $k = l + 1$. We have the following short exact sequence

$$0 \longrightarrow R/\left(x^h\right) \xrightarrow{\cdot x} R/\left(x^{h+i}\right) \longrightarrow R/\left(x\right) \longrightarrow 0$$

which induces a long exact sequence

$$\cdots \longrightarrow H^{i-1}_{\mathfrak{m}}\left(R/\left(x^{h+i}\right)\right) \xrightarrow{\beta_1} H^{i-1}_{\mathfrak{m}}\left(R/\left(x\right)\right) \xrightarrow{\delta} H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \xrightarrow{\beta_2} H^i_{\mathfrak{m}}\left(R/\left(x^{h+1}\right)\right) \longrightarrow \cdots$$

Note that x is a surjective element, hence β_1 is surjective, and so δ is the zero map. We conclude β_2 is injective.

Now, suppose

$$H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^k\right)\right)$$

is an injection for $k = h + n$. Set $h' = h + n$. By induction hypothesis we have that

$$H^i_{\mathfrak{m}}\left(R/\left(x^{h'}\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^{k+1}\right)\right)$$

is injective.

Suppose that 2 holds. Consider the short exact sequence

$$0 \longrightarrow R/\left(x^h\right) \xrightarrow{x^{k-h}} R/\left(x^k\right) \longrightarrow R/\left(x^{k-h}\right) \longrightarrow 0$$

By 2, we have that

$$H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^k\right)\right)$$

is injective for every $i \geq 0$. Hence for every $i \geq 0$

$$0 \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^k\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^{k-h}\right)\right) \longrightarrow 0$$

Finally, suppose 3 holds. Taking $k = h + 1$ we have that

$$0 \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^{h+1}\right)\right) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x\right)\right) \longrightarrow 0$$

Hence we conclude x is a surjective element. ∎

Proposition 7.1.3. *The following are equivalent:*

1. *x is a surjective element.*
2. *The multiplication map*
$$H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R)$$
is surjective for all $i \geq 0$.

Proof. By Proposition 7.1.2, the element x is surjective if and only if for every $h, k \in \mathbb{N}$ with $0 < h \leq k$ the map
$$H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \xrightarrow{\cdot x^{k-h}} H^i_{\mathfrak{m}}\left(R/\left(x^k\right)\right)$$
is injective. This is equivalent to have the direct limit system consisting on $\left(\left\{H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right)\right\}_{h\geq 1}, \cdot x^{k-h}\right)$ such that
$$\Phi_h : H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \to \varinjlim_h H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right),$$
is injective for every $h \in \mathbb{N}$ and every $i \in \mathbb{N}$. By the work of Horiuchi, Miller and Shimomoto [[HMS14], Lemma 2.2],
$$\varinjlim_h H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \cong H^i_{\mathfrak{m}}\left(H^1_{(x)}(R)\right) \cong H^{i+1}_{\mathfrak{m}}(R).$$
Consider the short exact sequence
$$0 \longrightarrow R \xrightarrow{\cdot x^h} R \longrightarrow R/\left(x^h\right) \longrightarrow 0$$
This induces a long exact sequence
$$\cdots \longrightarrow H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}\left(R/\left(x^h\right)\right) \longrightarrow H^{i+1}_{\mathfrak{m}}(R) \longrightarrow H^{i+1}_{\mathfrak{m}}(R) \longrightarrow \cdots$$
Since each Φ_h is injective, we are done. ■

Proposition 7.1.4. *Let x be a regular element. If $R/(x)$ is full, then x is a surjective element. In particular, if $R/(x)$ is F-anti-nilpotent, then x is a surjective element.*

Proof. Consider the map
$$\begin{aligned}\tilde{F} : R/(x) &\to R/\left(x^{p^e}\right)\\ [r] &\mapsto [F(r)]\end{aligned}$$
Note that for any $[r] \in R/(x)$, we have that
$$\begin{aligned}\pi \circ \tilde{F}([r]) &= \pi\,[F(r)]\\ &= [F(r)]\\ &= F([r]),\end{aligned}$$

where $\pi : R/\left(x^{p^e}\right) \to R/(x)$. This is, the following diagram commutes

$$\begin{array}{ccc} R/(x) & \xrightarrow{F} & R/(x) \\ & {\scriptstyle \tilde{F}}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & R/\left(x^{p^e}\right) & \end{array}$$

which induces the following commutative diagram

$$\begin{array}{ccc} R/(x) \otimes H^i_{\mathfrak{m}}(R/(x)) & \xrightarrow{\varphi} & H^i_{\mathfrak{m}}(R/(x)) \\ & {\scriptstyle \psi}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & H^i_{\mathfrak{m}}\left(R/\left(x^{p^e}\right)\right) & \end{array}$$

where $\varphi([r] \otimes v) = [r]\,F(v)$ and $\psi([r] \otimes v) = [r]\,\tilde{F}(v)$. Furthermore, since $R/(x)$ is F-full, φ is surjective. Hence $\pi \circ \psi$ is surjective, and so is π. We conclude x is a surjective element. ■

These propositions allow us to construct a family of non F-full local rings.

Example 7.1.5. Let $(R, \mathfrak{m})$ be a local ring with finite length cohomology, this is, $H^i_{\mathfrak{m}}(R)$ has finite length for every $i < \dim R$. Let $x \in R$ be a regular element. We want to show that being F-full implies being Cohen-Macaulay. Suppose $R/(x)$ is F-full. By Proposition 7.1.4, x is a surjective element. Hence

$$H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^i_{\mathfrak{m}}(R)$$

is surjective for every $i \geq 0$. This is, we have a descending chain

$$\cdots \subsetneq x^3 H^i_{\mathfrak{m}}(R) \subsetneq x^2 H^i_{\mathfrak{m}}(R) \subsetneq x H^i_{\mathfrak{m}}(R) = H^i_{\mathfrak{m}}(R).$$

However, since R has finite length cohomology and x is a regular element, there exists $n \in \mathbb{N}$ such that x^n annihilates $H^i_{\mathfrak{m}}(R)$ for every $i < \dim R$. Again, since x is a regular element $H^i_{H^i_{\mathfrak{m}}(R)}(R) = 0$ for every $i < \dim R$. Therefore, R is Cohen-Macaulay.

We can now prove a different version of Proposition 3.3.6 taking now $R/(x)$ F-full.

Remark 7.1.6. Let $R/(x)$ be F-full. Then by 7.1.4 is a surjective element. Using the idea of the proof form Proposition 3.3.6,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\ 0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \end{array}$$

induces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{t-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^t_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^t_{\mathfrak{m}}(R) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{t-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^t_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^t_{\mathfrak{m}}(R) & \longrightarrow & \cdots \end{array}$$

Note that if for some i $H^i_{\mathfrak{m}}(R) \neq 0$ and finitely generated, then $H^i_{\mathfrak{m}}(R)$ has finite length. Thus, there exists $e \gg 0$ such that $\mathfrak{m}^{p^e-1}H^i_{\mathfrak{m}}(R) = 0$, and so $x^{p^e-1}F^e = 0$. Hence $F^e = 0$ on $H^{t-1}_{\mathfrak{m}}(R/(x))$ for some $e \gg 0$. Since $R/(x)$ is F-full,

$$0 = H^{t-1}_{\mathfrak{m}}(R/(x)) = F^e\left(H^{t-1}_{\mathfrak{m}}(R/(x))\right)R.$$

Therefore, $\cdot x$ is an injection, which is a contradiction since $\sup\left(H^t_{\mathfrak{m}}(R)\right) = \{\mathfrak{m}\}$. We conclude $\operatorname{depth} R = f_{\mathfrak{m}}(R)$.

Proposition 7.1.7. *Let $x \in R$ be a regular element, and let s be a positive integer such that*

$$H^{s-1}_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^{s-1}_{\mathfrak{m}}(R/x)$$

is surjective and the Frobenius action on $H^{s-1}_{\mathfrak{m}}(R/(x))$ is injective. Then the map

$$H^s_{\mathfrak{m}}(R) \xrightarrow{x^{p-1}F} H^s_{\mathfrak{m}}(R)$$

is injective

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\ 0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \end{array}$$

Since $\cdot x$ is surjective in $H^{s-1}_{\mathfrak{m}}(R)$, we have the induced commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{s-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^s_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^s_{\mathfrak{m}}(R) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \\ 0 & \longrightarrow & H^{s-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^s_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^s_{\mathfrak{m}}(R) & \longrightarrow & \cdots \end{array}$$

Take $y \in \operatorname{Ker}\left(x^{p-1}F\right) \cap \operatorname{Soc}\left(H^s_{\mathfrak{m}}(R)\right)$. Since x is a regular element, $x \in \mathfrak{m}$, and so $xy = 0$. Hence, there exists $z \in H^{s-1}_{H^s_{\mathfrak{m}}(R)}(R/(x))$ such that $\alpha(z) = y$. Furthermore,

$$\begin{aligned}(\alpha \circ F)(z) &= x^{p-1}F(\alpha(z)) \\ &= x^{p-1}F(y) \\ &= 0\end{aligned}$$

Since F and α are both injective, we have that $z = y = 0$. We conclude $x^{p-1}F$ is injective. ■

Corollary 7.1.8. *For $x \in R$ a regular element, if $R/(x)$ is F-full, then R is F injective.*

Proof. The proof follows from Proposition 7.1.4 and Proposition 7.1.7. ■

7.2 Deformation of F-full and F-anti-nilpotent singularities

Our goal in this section is to prove that given an element x in R, if $R/(x)$ is either F-anti-nilpotent or F-full, then so is R.

Lemma 7.2.1. *Let x be a surjective element of R. Let $N \subseteq H^i_{\mathfrak{m}}$ be an F-stable submodule. Let $L = \cap_{t\in\mathbb{N}} x^t N$. Then L is an F-stable submodule of $H^i_{\mathfrak{m}}(R)$ and we have the following commutative diagram for every $e \geq 1$*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\
0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0
\end{array}$$

where $\phi : H^{i-1}_{\mathfrak{m}}(R/(x)) \to H^i_{\mathfrak{m}}(R)$.

Proof. Since x is a surjective element, by Proposition 7.1.3 the map $x : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is surjective for all $i > 0$. Consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle F^e} & & \\
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0
\end{array}$$

which induces the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{s-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^s_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^s_{\mathfrak{m}}(R) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\
0 & \longrightarrow & H^{s-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^s_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^s_{\mathfrak{m}}(R) & \longrightarrow & \cdots
\end{array}$$

where the rows are short exact sequences since the map x is surjective. Therefore to prove the lemma it suffices to prove that L is F-stable and

$$0 \longrightarrow H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) \xrightarrow{\phi} H^i_{\mathfrak{m}}(R)/L \xrightarrow{\cdot x} H^i_{\mathfrak{m}}(R)/L \longrightarrow 0$$

is exact. Let $n \in N$, then since N is F-stable

$$\begin{aligned} F^e\left(x^t n\right) &= x^{tp^e} n \\ &= x^t\left(x^{t(p^e-1)} n\right) \\ &\in x^t N. \end{aligned}$$

Hence L is F-stable. Now note that

$$\operatorname{Im}(\phi) = \ker(\cdot x) = 0 :_{H_{\mathfrak{m}}(R)} x$$

Therefore $L + \operatorname{Im}(\phi) \subseteq L :_{H_{\mathfrak{m}}(R)} x$

On the other hand, let $y \in L :_{H_{\mathfrak{m}}(R)} x$. Then $xy \in L$. Since $L = xL$, there exists $z \in L$ such that $yx = zx$. Hence $x(y - z) = 0$, and so $y - z \in \operatorname{Im}(\phi)$. Finally $y = z + y - z \in L + \operatorname{Im}(\phi)$. Thus $L + \operatorname{Im}(\phi) = L :_{H_{\mathfrak{m}}(R)} x$. We conclude the exactness of the sequence. ∎

Theorem 7.2.2 ([MQ18])**.** *Let $x \in R$ be a regular element. The we have*

1. *If $R/(x)$ is F-anti-nilpotent, then so is R.*
2. *If $R/(x)$ is F-full, then so is R.*

Proof.

1. Let $N \subseteq H^i_{\mathfrak{m}}(R)$ be an F-stable submodule. Since $R/(x)$ is F-anti-nilpotent, by Proposition 7.1.4, x is a surjective element. Let $L = \cap_{t\in\mathbb{N}} x^t N$. By Lemma 7.2.1, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \end{array}$$

   Since ϕ is a map of $R\{F\}$-modules, we have that ϕ and F^e commutes and so L is F-stable.

   We claim the map $x^{p^e-1}F^e$ is injective. Take $y \in \operatorname{Ker}\left(x^{p^e-1}F^e\right) \cap \operatorname{Soc}\left(H^i_{\mathfrak{m}}(R)/L\right)$. Then $xy = 0$, and so there exists $z \in H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$ such that $\phi(z) = y$. Furthermore $F^e(z) = 0$. Since F is injective on $H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$, so is F^e, and so $z = y = 0$.

   Now, we have the descending chain

$$\cdots \subseteq x^3 N \subseteq x^2 N \subseteq N \subset N.$$

   Since $H^i_{\mathfrak{m}}(R)$ is Artinian, $\cap_{t\in\mathbb{N}} x^t N = x^n N$ for some $n \gg 0$. Hence for $e \gg 0$

$$x^{p^e-1}(N) \subseteq x^{p^e-1}N = L.$$

   Since $x^{p^e-1}F$ is injective, $N \subseteq L$, and so $N = L$. Finally, let $r \in H^i_{\mathfrak{m}}(R)/L$ be such that $F(r) = 0$. Therefore $x^{p^e-1}F^e(r) = 0$ and so $r = 0$. We conclude R is F-anti-nilpotent.

2. Let $N = F\left(H^i_{\mathfrak{m}}(R)\right)R$. Then N is F-stable. Since $R/(x)$ is F- full, by Proposition 7.1.4, x is a surjective element. Let $L = \cap_{t\in\mathbb{N}} x^t N$. By Lemma 7.2.1, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \end{array}$$

   Furthermore, the descending chain

$$\cdots \subseteq x^3 N \subseteq x^2 N \subseteq N \subset N$$

   stabilizes for some $n \gg 0$ since $H^i_{\mathfrak{m}}(R)$ is Artinian. Hence $L = x^n N$. Let $y \in H^i_{\mathfrak{m}}(R)$. Then $F^e(y) \in N$, and so $x^{p^e-1}F^e(y) \in L$ for some $e \gg 0$. Since x is a regular element $x^{p^e-1}F^e = 0$ for $e \gg 0$. By the diagram above

$$H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) \hookrightarrow H^i_{\mathfrak{m}}(R)/L,$$

and so $F^e = 0$ in $H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$ for $e \gg 0$. Hence F is nilpotent on $H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$, and so $F^e\left(H^{i-1}_{\mathfrak{m}}(R/(x))\right) \subseteq \phi^{-1}(L)$ for $e \gg 0$. Thus $F^e\left(H^{i-1}_{\mathfrak{m}}(R/(x))\right) R \subseteq \phi^{-1}(L)$ for $e \gg 0$. Since $R/(x)$ is F-full, $F^e\left(H^{i-1}_{\mathfrak{m}}(R/(x))\right) = \phi^{-1}(L)$ for $e \gg 0$. Therefore, $x : H^i_{\mathfrak{m}}(R)/L \to H^i_{\mathfrak{m}}(R)/L$ is an isomorphism, which is impossible unless $H^i_{\mathfrak{m}}(R) = L$. Otherwise, for any $y \in \operatorname{Soc}\left(H^i_{\mathfrak{m}}(R)/L\right) \setminus \{0\}$, we have that $xy = 0$. We conclude $H^i_{\mathfrak{m}}(R) = N$, and so R is F-full.

■

7.3 Deformation of F-injectivity

In this section we prove that given an element $x \in R$, if $R/(x)$ is F-injective, then so is R under certain conditions. In order to do this we define strictly filter regular sequences and a new singularity called strongly F-injectivity.

Definition 7.3.1. A regular element x is called a *strictly filter regular element* if

$$\operatorname{Coker}\left(H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^i_{\mathfrak{m}}(R)\right)$$

has finite length for all $i \geq 0$.

Lemma 7.3.2. *Suppose $K = R/\mathfrak{m}$ is perfect. Let M be an R-module with an injective Frobenius action F. Suppose $L \subseteq M$ is an F-stable submodule of finite length. Then the induced Frobenius action on M/L is injective.*

Proof. Let $x \in L$. Then, since L has finite length

$$F^e(\mathfrak{m}x) = m^{[p^e]} = 0$$

for $e \gg 0$. This implies that $\mathfrak{m}x = 0$. Hence we have a Frobenius action F on L a K-vector space. Let $L' = F(L)$. Note that $L' \subseteq L$ is a K^p-vector subspace. Since F is injective, $\dim_{K^p} L' = \dim_K L$. However, $K = K^p$, and so $L' = L$. This implies that F is surjective, hence bijective. Finally, if $x \notin L$, then $F(x) \notin L$. We conclude that $F : M/L \to M/L$ is injective. ■

The following example emphasizes the importance of K to be perfect in Lemma 7.3.2.

Example 7.3.3. Let $A = \mathbb{F}_p[t]$, and $R = K = \mathbb{F}_p(t)$. Consider the A-module $Ae_1 \oplus Ae_2$ with the Frobenius action

$$\begin{aligned} F : Ae_1 \oplus Ae_2 &\to Ae_1 \oplus Ae_2 \\ (f(t), g(t)) &\mapsto (f(t)^p + tg(t)^p, 0). \end{aligned}$$

This action is injective, since

$$\begin{aligned} F(f(t), g(t)) = 0 &\Leftrightarrow f(t)^p + tg(t)^p = 0 \\ &\Leftrightarrow f(t)^p = 0 = g(t)^p \\ &\Leftrightarrow f(t) = 0 = g(t). \end{aligned}$$

Moreover $Ae_1 \oplus 0 \subseteq Ae_1 \oplus Ae_2$ is F-stable. Hence,

$$F\left(\frac{Ae_1 \oplus Ae_2}{Ae_1 \oplus 0}\right) = 0.$$

Now, localizing at t we get a Frobenius action on the modules $M = Ke_1 \oplus Ke_2$ and $L = Ke_1 \oplus 0$. Note $L \subseteq M$ is a F-stable of finite length, since $mL = 0$. However the induced Frobenius action on M/L is not injective since F-stable. Hence,

$$F\left(M/L\right) = F\left(\frac{Ke_1 \oplus Ke_2}{Ke_1 \oplus 0}\right) = 0.$$

Theorem 7.3.4. *Suppose $K = R/\mathfrak{m}$ is perfect. Let $x \in R$ be a strictly filter regular element. If $R/(x)$ is F-injective, then the map $x^{p-1}F : H^i_{\mathfrak{m}}\left(R\right) \to H^i_{\mathfrak{m}}\left(R\right)$ is injective for every $i \geq 0$. In particular, R is F-injective.*

Proof. Let $L_i = \operatorname{Coker}\left(H^i_{\mathfrak{m}}\left(R\right) \xrightarrow{\cdot x} H^i_{\mathfrak{m}}\left(R\right)\right)$. Since x is a strictly filter regular element, L_i has finite length for all $i \geq 0$. The following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/(x) & \longrightarrow & 0
\end{array}$$

induces the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & L_{i-1} & \longrightarrow & H^{i-1}_{\mathfrak{m}}\left(R/(x)\right) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}\left(R\right) & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}\left(R\right) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & L_{i-1} & \longrightarrow & H^{i-1}_{\mathfrak{m}}\left(R/(x)\right) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}\left(R\right) & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}\left(R\right) & \longrightarrow & \cdots
\end{array}$$

Therefore, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}\left(R/(x)\right)/L_{i-1} & \xrightarrow{\alpha} & H^i_{\mathfrak{m}}\left(R\right) & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}\left(R\right) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}\left(R/(x)\right)/L_{i-1} & \xrightarrow{\alpha} & H^i_{\mathfrak{m}}\left(R\right) & \xrightarrow{\cdot x} & H^i_{\mathfrak{m}}\left(R\right) & \longrightarrow & \cdots
\end{array}$$

By Lemma 7.3.2, the Frobenius action on $H^{i-1}_{\mathfrak{m}}\left(R/(x)\right)/L_{i-1}$ is injective. Now, we want to show that $x^{p-1}F$ is injective. Let $y \in \operatorname{Ker}\left(x^{p-1}F\right) \cap \operatorname{Soc}\left(H^i_{\mathfrak{m}}\left(R\right)\right)$. Since x is a regular element, $x \in \mathfrak{m}$, and so $xy = 0$. Thus, there exists $z \in H^{i-1}_{\mathfrak{m}}\left(R/(x)\right)/L_{i-1}$ such that $\alpha\left(z\right) = y$. Furthermore

$$\begin{aligned}
\left(\alpha \circ F\right)\left(z\right) &= x^{p-1}F\left(\alpha\left(z\right)\right) \\
&= x^{p-1}F\left(y\right) \\
&= 0
\end{aligned}$$

Since F and α are both injective, we have that $z = y = 0$. We conclude $x^{p-1}F$ is injective. ■

Proposition 7.3.5. *Suppose $K = R/\mathfrak{m}$ is perfect. Let $x \in R$ be a regular element such that $R/(x)$ is F-injective. Let s be a positive integer such that $H_{\mathfrak{m}}^{s-1}(R/(x))$ has finite length. Then the map*

$$x^{p-1}F : H_{\mathfrak{m}}^{s+1}(R/) \to H_{\mathfrak{m}}^{s+1}(R/)$$

is injective.

Proof. Given the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow R/(x) \longrightarrow 0$$

we have the long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{m}}^{s-1}(R/(x)) \longrightarrow H_{\mathfrak{m}}^{s}(R) \xrightarrow{\cdot x} H_{\mathfrak{m}}^{s}(R) \longrightarrow H_{\mathfrak{m}}^{s}(R/(x)) \longrightarrow H_{\mathfrak{m}}^{s+1}(R) \longrightarrow \cdots$$

Since $H_{\mathfrak{m}}^{s-1}(R/(x))$ has finite length, then $\operatorname{Ker}(\cdot x)$ has finite length too and $\operatorname{Supp}(\operatorname{Ker}(\cdot x)) = \{\mathfrak{m}\}$. We want to show that $L_s = \operatorname{Coker}(\cdot x)$ has finite length.

Without loss of generality we can assume R is complete. Since $\operatorname{Ker}(\cdot x)$ has finite length and $H_{\mathfrak{m}}^{s}(R)^{\vee}$ is finitely generated as R-module, then the map

$$H_{\mathfrak{m}}^{s}(R)^{\vee} \xrightarrow{\cdot x} H_{\mathfrak{m}}^{s}(R)^{\vee}$$

is surjective when localizing at any prime ideal but $\mathfrak{m}$. Then $\cdot x$ is an isomorphism on $H_{\mathfrak{m}}^{s}(R)^{\vee}$, as $H_{\mathfrak{m}}^{s}(R)^{\vee}$ is module finite. Therefore the $\operatorname{Ker}\left(H_{\mathfrak{m}}^{s}(R)^{\vee} \xrightarrow{\cdot x} H_{\mathfrak{m}}^{s}(R)^{\vee} \right)$ has finite length. Dualizing we have that $\operatorname{Coker}\left(H_{\mathfrak{m}}^{s}(R) \xrightarrow{\cdot x} H_{\mathfrak{m}}^{s}(R) \right)$ has finite length too. Using the same argument as in Theorem 7.3.4, we get that $x^{p-1}F$ is injective. ■

Corollary 7.3.6. *Suppose $K = R/\mathfrak{m}$ is perfect. Let $x \in R$ be a regular element such that $R/(x)$ is F-injective. Then the map $x^{p-1}F : H_{\mathfrak{m}}^{i}(R) \to H_{\mathfrak{m}}^{i}(R)$ is injective for all $i \leq f_{\mathfrak{m}}(R/(x)) + 1$. In particular, if $R/(x)$ is generalized Cohen-Macaulay, then R is F-injective.*

Proof. This follows from Proposition 7.3.5 changing s for i, for $i < f_{\mathfrak{m}}(R/(x))$. ■

Definition 7.3.7. We say that R is *strongly F-injective* is R is F-injective and F-full.

Remark 7.3.8. By Remark 3.2.23 and Remark 3.3.2, we have that F-anti-nilpotent implies strongly F-injective, which implies F-injective. Furthermore, if R is Cohen-Macaulay and F-injective, then R strongly F-injective.

Corollary 7.3.9 ([MQ18])**.** *Let $x \in R$ be a regular element. If $R/(x)$ is strongly F-injective, then so is R.*

Proof. First, by Theorem 7.2.2, R is F-full and by Corollary 7.1.8 R is F-injective. We conclude R is strongly F-injective. ■

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